University of Oslo / Department of Economics / NCF

ECON3120/4120 Mathematics 2 – on the 2013–06–04 exam (draft)

- This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading process.
- Weighting: assigned at the grading committee's discretion. (In case of appeals: the new grading committee assigns weighting at their discretion.) The problem set was written with the intention that a uniform weighting over letter-enumerated items should be a possible choice.

Problem 1 Define for each real number t the matrix $\mathbf{A}_t$ and the vector $\mathbf{b}_t$ by

$$\mathbf{A}_t = \begin{pmatrix} t & 3 & 0 \\ 1 & t+1 & 3 \\ 0 & 1 & t \end{pmatrix} \qquad \text{and} \qquad \mathbf{b}_t = \begin{pmatrix} t \\ t^2 \\ t^3 \end{pmatrix}$$

(a) (i) Find real numbers p, q such that $\mathbf{A}_s + \mathbf{A}_t = p\mathbf{A}_q$.

(ii) Calculate the determinant of $\mathbf{A}_t$.

(b) Find those t for which there is a solution (one or more!) of the equation system

$$\mathbf{A}_t\,\mathbf{x} = \mathbf{b}_t$$

(where $\mathbf{x} = (x_1, x_2, x_3)'$ is the unknown vector.)

(c) Is there any t such that the equation system

$$\mathbf{A}_t^{2013}\,\mathbf{x} = \mathbf{b}_t$$

has *infinitely* many solutions? ($\mathbf{A}_t^{2013}$ denotes the 2013th power.)

On the solution:

(a) (i) The off-diagonal elements match if and only if $\underline{\underline{p = 2}}$. With $p = 2$, put $\underline{\underline{q = (s+t)/2}}$ and the diagonal elements match too.

(ii) Cofactor expansion, e.g. by first column, yields (notice the minus in front of the 1): $t \cdot [t(t+1) - 3] - 1 \cdot [3t - 0] = \underline{\underline{t^3 + t^2 - 6t}}$.

(b) There is a solution when the determinant is nonzero, and also (with $\mathbf{x} = \mathbf{0}$) when $t = 0$. The other zeroes of the determinant are when $t^2 + t - 6 = 0$ i.e. $t = -3$, $t = 2$.
For the case $t = -3$, Gaussian elimination yields

$$\begin{pmatrix} -3 & 3 & 0 & \vdots & -3 \\ 1 & -2 & 3 & \vdots & 9 \\ 0 & 1 & -3 & \vdots & -27 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & 0 & \vdots & -1 \\ 0 & -1 & 3 & \vdots & 8 \\ 0 & 1 & -3 & \vdots & -27 \end{pmatrix}$$

by scaling the first by 1/3 and then adding it to the second. Add the second and third for a contradiction.
For the case $t = 2$:

$$\begin{pmatrix} 2 & 3 & 0 & \vdots & 2 \\ 1 & 3 & 3 & \vdots & 4 \\ 0 & 1 & 2 & \vdots & 8 \end{pmatrix}$$

Subtract twice the second from the first to get the first row: $(0 \;\; -3 \;\; -6 \, \vdots \, -6)$ which contradicts the third.
So the system has solution when $\underline{\underline{t \notin \{-3, 2\}}}$.

(c) $\underline{\underline{\text{Yes, for } t = 0}}$: $|\mathbf{A}_t^{2013}| = |\mathbf{A}_t|^{2013}$, so we must necessarily have $|\mathbf{A}_t| = 0$. When $t = 0$ we have zero determinant, and we have at least one solution (the null) and thus infinitely many.

Problem 2 Consider the function $F(t) = t^t - \frac{1}{\sqrt[3]{3}}$ defined for $t > 0$.

(a) Show that

$$\int t^t(1 + \ln t)\, dt = F(t) + C$$

and use this to find

$$\int_0^{\ln 2} (z + 1)e^{z(1+e^z)}\, dz.$$

(*Hint:* for the latter, use the substitution $t = e^z$.)

(b) How many zeroes does F have?
(You are not asked to calculate any, but observe that $F(1/3) = 0$ and that $F'(1/3) < 0$.)

(c) Use part (a) to find the particular solution which passes through $(t_0, x_0) = (2, 2)$ of the differential equation

$$3\dot{x}(t) = \frac{t^t(1 + \ln t)}{\big(x(t)\big)^2}$$

On the solution:

(a) For the first part: $F'(t) = (t^t)'$ which has been covered in the lecture: solve either as u^v by the chain rule, or from $t^t = e^{t \ln t}$ or by the formula $G'(t) = G(t)(\ln G(t))'$. In all cases, we end up at $F'(t) = t^t(1 + \ln t)$ which is the integrand.
With $t = e^z$ we have $dt = e^z dz$, taking out the «1» in the exponent:

$$\int (z + 1)e^{z(1+e^z)}\, dz = \int (\ln t + 1)t^t dt = t^t + C = e^{ze^z} + C$$

so that the definite integral is $2^2 - e^0 = \underline{\underline{3}}$.
Note: The students should know that if they substitute in definite integrals, they must substitute the limits too.

(b) We have that F' has a zero for $t = 1/e$, is < 0 on $(0, 1/e)$ and > 0 on $(1/e, \infty)$. So there is *at most* one zero in each of these intervals. In $(0, 1/e)$ there is one, namely for $t = 1/3$. In $(1/e, \infty)$ there is also one, by the intermediate value theorem; as F decreases from zero at $t = 1/3$ to $t = 1/e$, we have $F(1/e) < 0$ – and $F \to +\infty$ as $t \to +\infty$. So the answer is $\underline{\underline{\text{two zeroes}}}$.

(c) Separate: $3x^2\, dx = t^t(1 + \ln t)\, dt$ and integrate: $x^3 = t^t + C$ (using part (a)). Find $C = 2^3 - 2^2 = 4$. Solution: $\underline{\underline{x(t) = \sqrt[3]{t^t + 4}}}$

Problem 3 Let N be a positive integer (i.e. $1, 2, ...$) and let f be the function

$$f(x, y) = xy^{N+1} - (x+1)\ln(x+1)$$

defined for $x > -1$, all y.

(a) Find and classify the stationary points of f.

(b) Does f have any global maximum or minimum?

In the following, consider the problem

$$\max\ f(x, y) \qquad \text{subject to} \quad \begin{cases} x \geq 0 \\ y \geq 0 \\ y \leq 1 - x \end{cases} \tag{P}$$

(c) Decide whether this problem has a solution or not, and state the associated Kuhn–Tucker conditions.

(d) Put $x = 0$ in the Kuhn–Tucker conditions. For what $y \in [0, 1]$ will the Kuhn–Tucker conditions be satisfied for the point $(0, y)$?

(e) Show that the Kuhn–Tucker conditions cannot be satisfied when $0 < x < 1 - y$.

On the solution:

(a) We have

$$f'_x(x, y) = y^{N+1} - \ln(x+1) - 1$$
$$f'_y(x, y) = (N+1)xy^N$$

and $f'_y = 0$ if and only if $xy = 0$. For $x = 0$: stationary point $(x, y) = \underline{\underline{(0, 1)}}$.

For $y = 0$: stationary point where $\ln(x+1) = -1$, i.e. $(x, y) = \underline{\underline{(\frac{1}{e} - 1, 0)}}$.

To classify, we have

$$f'_{xx}(x, y) = -\frac{1}{x+1} \qquad (< 0)$$
$$f'_{xy}(x, y) = (N+1)y^N$$
$$f'_{yy}(x, y) = N(N+1)xy^{N-1}$$

For $(x, y) = (0, 1)$, the latter vanishes and the Hessian is negative: <u>saddle point</u>. For $(x, y) = (\frac{1}{e} - 1, 0)$, the mixed derivative vanishes, and so does f''_{yy} except when $N = 1$ when it is negative: <u>local maximum for $N = 1$, no conclusion otherwise</u>.

Note: The above «no conclusion» answer is fully accepted as the candidates are only expected to classify with the standard tool of the course – that was clarified explicitely at the beginning of the exam.
The punctured-neighbourhood version cannot be required, and neither can the following argument (but it is allowed and can even replace the 2nd derivative test):

- For N even, f is strictly monotone in y for all $x \neq 0$, hence $(x, y) = (\frac{1}{e} - 1, 0)$ is a saddle point.
- For N odd and $x \leq 0$, we have $f(x, y) \leq f(x, 0) = -(x+1)\ln(x+1)$ with maximum for $x = \frac{1}{e} - 1$ (as $f''_{xx} < 0$). (This is not a global maximum, only over $x \leq 0$.)

(b) No: Fix an $x \neq 0$ and let $y \to +\infty$. Then $f \to +\infty \cdot \text{sign}(x)$.

(c) Yes, by the extreme value theorem: The constraints form a closed, bounded (nonempty) set, where f is continuous.
Define the Lagrangian $L(x, y) = f(x, y) - \lambda(x + y - 1) + \alpha x + \beta y$ (goes with/without explicitely rewriting the constraints as $-x \leq 0$, $-y \leq 0$, $x + y \leq 1$). Conditions:

$$y^{N+1} - \ln(x+1) - 1 = \lambda - \alpha \tag{1}$$
$$(N+1)xy^N = \lambda - \beta \tag{2}$$
$$\lambda \geq 0 \qquad \text{and } \lambda = 0 \text{ if } x + y < 1 \tag{3}$$
$$\alpha \geq 0 \qquad \text{and } \alpha = 0 \text{ if } x > 0 \tag{4}$$
$$\beta \geq 0 \qquad \text{and } \beta = 0 \text{ if } y > 0 \tag{5}$$

(In addition, the constraints must be satisfied; it is OK to include those in the «Kuhn–Tucker conditions» even though the book does not.)

(d) With $x = 0$, the stationarity conditions are

$$y^{N+1} - 1 = \lambda - \alpha$$
$$\beta = \lambda$$

and the Kuhn–Tucker conditions are satisfied with $\lambda = \beta = 0$ and $\alpha = 1 - y^{N+1} \geq 0$ for all $y \in [0, 1]$.
(*Note*: It is OK to just (guess and) state $\lambda = \beta = 0$ and then verify the nonnegativity for α. While it easily follows that we *must* have $\lambda = \beta = 0$, the candidates cannot be required to justify that claim, as it is easy to see that it fits once the suggested α is nonnegative.)

(e) When $x > 0$ and $x + y < 1$, we have $\alpha = \lambda = 0$. By (2), we must also have $\beta = 0$ as the left-hand side is nonnegative. Hence we must have a stationary point, and from part (a), the only admissible is $(0, 1)$ – which violates (even both inequalities of) the assumption $0 < x < 1 - y$.