University of Oslo / Department of Economics / NCF

ECON3120/4120 Mathematics 2 – on the 2013–12–10 exam (draft)

- This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading process.
- Weighting: assigned at the grading committee's discretion. (In case of appeals: the new grading committee assigns weighting at their discretion.) The problem set was written with the intention that a uniform weighting over letter-enumerated items should be a possible choice.

Problem 1 In this problem, let t be a constant and consider the linear equation system $\mathbf{A}_t\mathbf{x} = \mathbf{b}$ in the unknown vector $\mathbf{x}$, where

$$\mathbf{A}_t = \begin{pmatrix} 12-3t & 2 & 2 \\ 1 & 0 & 2 \\ 12 & 2-t & 2-t \end{pmatrix}, \qquad \text{and} \qquad \mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ 12 \end{pmatrix}.$$

(a) Find q such that $\mathbf{A}_t$ has determinant equal to $q \cdot t(t-6)$.

(b) For what values of the constant t will the equation system $\mathbf{A}_t\mathbf{x} = \mathbf{b}$ have (i) unique solution, (ii) no solution, (iii) several solutions?

On the solution:

(a) Cofactor expansion along the second row yields (note the negative signs!)

$$-1 \begin{vmatrix} 2 & 2 \\ 2-t & 2-t \end{vmatrix} - 2 \begin{vmatrix} 12-3t & 2 \\ 12 & 2-t \end{vmatrix} = 0 - 2(24 - 12t - 6t + 3t^2 - 24)$$
$$= -2t(3t-18) \quad \text{so that } \underline{\underline{q = -6}}.$$

(*Note:* it would simplify to subtract the first row from the third (or vice versa), or the second column from the third (or vice versa); in the latter case it immediately follows that only the mid-right cofactor contributes, but this also follows from the vanishing first term above.)

(b)
- From (a) it follows that we have unique solution if and only if $q \notin \{0, 6\}$.
- For the case $t = 0$, the first and third equation contradict, i.e. no solution.
- Also for the case $t = 6$, the first and third equation contradict (add twice the first to the third). No solution.

Problem 2 Let $p > 0$ be a constant and let $g(x) = (2x + p)e^{-px} + pxe^{-2px}$.

(a) (i) Find $\lim_{x\to-\infty} g(x)$ or show that it does not exist.

(ii) Show that $\lim_{x\to+\infty} g(x) = 0$.

(b) Show that there exists an $\hat{x} < 0$ such that $g(\hat{x}) = 0$.

(c) Find $\int g(x)\,dx$ and decide whether $\int_p^\infty g(x)\,dx$ converges.

On the solution:

(a) (i) $p > 0$, so as $x \to -\infty$ both terms tend to $(-\infty)\cdot(+\infty) = \underline{\underline{-\infty}}$.

(ii) As $x \to +\infty$, both terms are polynomial growth times exponential decay, and therefore tend to 0.

(b) From (a)(i), $g < 0$ for all sufficiently negative x. On the other hand $g(0) = pe^0 + 0 > 0$, so the intermediate value theorem grants at zero $\hat{x} < 0$.

(c) Integrating by parts we have $\int xe^{-ax}\,dx = D - e^{-ax}(ax+1)/a^2$ for any $a \neq 0$; and, $\int pe^{-px}\,dx = C - e^{-px}$. Inserting, we get

$$\int g(x)\,dx = \underline{\underline{K - e^{-px} - \frac{2}{p^2}e^{-px}(px+1) - \frac{1}{4p}e^{-2px}(2px+1)}}$$

$$\text{or, if you like} \quad = K - [\frac{2x}{p} + \frac{p^2+2}{p^2}]e^{-px} - [\frac{1}{2}x + \frac{1}{4p}]e^{-2x}$$

For the improper integral $\int_p^\infty g(x)\,dx$, which by definition equals $\lim_{R\to+\infty}\int_p^R g(x)\,dx$, the lower limit of integration is no issue; the question is whether

$$-\lim_{x\to\infty}\{e^{-px} + \frac{2}{p^2}e^{-px}(px+1) + \frac{1}{4p}e^{-2px}(2px+1)\}$$

converges. It <u>does converge</u>, as each term is polynomial growth times exponential decay.

Problem 3 Let r be a positive constant. Consider the problem

$$\text{minimize} \quad e^{2y+(1-2r)x-r} - \ln(x+y+r/2) \qquad \text{subject to} \quad x+2y=r$$

Write out the associated Lagrange conditions, and show that they are equivalent to

$$2r(x+2r)e^{-2rx} + 1 = 0.$$

(Hint: Eliminate the multiplier and then y.)

On the solution: The Lagrange conditions are

$$(1-2r)e^{2y+(1-2r)x-r} - \frac{1}{x+y+r/2} = \lambda \tag{1}$$

$$2e^{2y+(1-2r)x-r} - \frac{1}{x+y+r/2} = 2\lambda \tag{2}$$

$$x+2y=r \tag{3}$$

Subtracting twice of (1) from (2), we get

$$0 = (2-2+4r)e^{2y+(1-2r)x-r} - (1-2)\frac{1}{x+y+r/2}$$

and by inserting $y=(r-x)/2$ the right-hand side becomes

$$= 4re^{r-x+x-2rx-r} + \frac{1}{x+r/2-x/2+r/2} = \frac{1}{r+x/2}\Big[(4r^2+2rx)e^{-2rx}+1\Big]$$

Problem 4

(a) (i) Show that

$$\int \Big[2t(\ln t - 1) - \ln t\Big] e^{-2t}\,\mathrm{d}t = t(1 - \ln t)e^{-2t} + C$$

(ii) Show that $t \cdot (1 - \ln t)$ is a particular solution of the differential equation

$$\dot{x}(t) = 2x(t) + 2t(\ln t - 1) - \ln t, \qquad t > 0 \qquad (*)$$

(b) Find the general solution of the differential equation $(*)$. (Hint: Use part (a).)

(c) One particular solution of $(*)$ passes through the point where $(t, x) = (1, 3)$. Find the equation for the tangent line at that point.

On the solution:

(a) These follow by (i) differentiating the right-hand side, and (ii) calculating the left-hand and right-hand sides and verifying.

(b) By formula or by integrating factor,

$$\frac{d}{dt}\big(xe^{-2t}\big) = (\dot{x} - 2x)e^{-2t} = \big[2t(\ln t - 1) - \ln t\big]e^{-2t}$$

and using (a)(i), $xe^{-2t} = t(1 - \ln t)e^{-2t} + C$ so that

$$\underline{\underline{x(t) = t(1 - \ln t) + Ce^{2t}}}$$

(c) For this part, we do not need the general solution. The line has equation $x - x_0 = \dot{x}(t_0)(t - t_0)$, and $\dot{x}(1) = 2\cdot 3 + 2\cdot 1(\ln 1 - 1) - \ln 1 = 4$ from the differential equation. Therefore, the answer is $x = 3 + 4(t - 1) = \underline{\underline{4t - 1}}$.

Problem 5 An exam problem in ECON5155 in December 2012, led to an equation system of the form

$$\begin{aligned}(e^{tu} + e^{-tu})v - u + s &= 0\\ (e^{tu} - e^{-tu})tv &= 1\end{aligned}$$

which defines u and v as continuously differentiable functions of s and t around the point P where $s = -2(e+3)/(e-1)$, $t = 1/4$, $u = 2$ and $v = 4\sqrt{e}/(e-1)$. (You are not supposed to show this.)

(a) Differentiate the equation system (i.e. calculate differentials).

(b) Use the differentiated system to find the value of $\partial u/\partial s$ at P.
(You *are required to* use the differentiated system, even though there might be other methods.)

On the solution:

(a) Differentiation yields e.g. the form

$$\begin{aligned} ds + uv(e^{tu} - e^{-tu})\,dt + \big[tv(e^{tu} - e^{-tu}) - 1\big]\,du + (e^{tu} + e^{-tu})\,dv &= 0\\ \big[tu(e^{tu} + e^{-tu}) + e^{tu} - e^{-tu}\big]v\,dt + t^2v(e^{tu} + e^{-tu})\,du + t(e^{tu} - e^{-tu})\,dv &= 0\end{aligned}$$

(b) At P, we can insert the coordinates; note in particular that $e^{tu} \pm e^{-tu} = e^{-1/2}(e \pm 1)$. Furthermore, since we are only interesting in partial changes wrt. s we can put $dt = 0$, in which case we obtain

$$\begin{aligned} 1 + \overbrace{\Big[\frac{e^{1/2}}{e-1}e^{-1/2}(e-1) - 1\Big]}^{=0}\,\frac{\partial u}{\partial s} + e^{-1/2}(e+1)\,\frac{\partial v}{\partial s} &= 0\\ \frac{1}{4}\cdot\frac{e^{1/2}}{e-1}\cdot e^{-1/2}(e+1)\,\frac{\partial u}{\partial s} + \frac{1}{4}\cdot\frac{e-1}{e^{1/2}}\,\frac{\partial v}{\partial s} &= 0\end{aligned}$$

so that

$$\frac{\partial u}{\partial s} = \frac{(e-1)^2}{(e+1)e^{1/2}}\cdot\Big(-\frac{\partial v}{\partial s}\Big) = \frac{(e-1)^2}{(e+1)e^{1/2}}\cdot\frac{e^{1/2}}{e+1} = \underline{\underline{\Big(\frac{e-1}{e+1}\Big)^2}}$$