2014-03-12 NCF

ECON 3120 / 4120 —
the 2014-05-23 exam

* We did not produce a solution note or grading guideline spring 2014. (Addendum: actually I noticed only afterwards that Espen Stokkereit did produce one. - NCF)

* This note was written for the review lecture 2014-12-03.

* In contrast to the (also handwritten) note to the 2011-12-15 exam, which was "minimal" (2 pages) this one is closer to an actual exam paper. However it does still employ "fairly efficient" approaches.
  - → I do recommend "new problem? new sheet of paper!"; I deviated for 1(b) and 1(c) as they have so much in common.
  - → There <u>are</u> some remarks that do not belong in an exam paper. (Should be easy to tell!)

1 (a)

(i) $\bar{S}_\epsilon = \epsilon\bar{A} + M_\epsilon = \begin{pmatrix} \epsilon & 2\epsilon+1 & \epsilon-1 \\ 2\epsilon+2\epsilon & \epsilon-\epsilon & 2\epsilon+0 \\ \epsilon-\epsilon & \epsilon-\epsilon & 2\epsilon+1 \end{pmatrix}$

$$= \underline{\underline{\begin{pmatrix} \epsilon & 2\epsilon+1 & \epsilon-1 \\ 4\epsilon & 0 & 2\epsilon \\ 0 & 0 & 2\epsilon+1 \end{pmatrix}}}$$

(ii) $\bar{P}_\epsilon = \bar{A}\bar{M}_\epsilon = \begin{pmatrix} 4\epsilon-\epsilon & 1-2\epsilon-\epsilon & -1+1 \\ 2\epsilon-2\epsilon & 2-\epsilon-2\epsilon & -2+2 \\ 2\epsilon-2\epsilon & 1-\epsilon-2\epsilon & -1+2 \end{pmatrix}$

$$= \underline{\underline{\begin{pmatrix} 3\epsilon & 1-3\epsilon & 0 \\ 0 & 2-3\epsilon & 0 \\ 0 & 1-3\epsilon & 1 \end{pmatrix}}}$$

(iii) $\bar{Q}_\epsilon$ has an inverse if and only if

$|\bar{Q}_\epsilon| \neq 0$ i.e. $\Leftrightarrow$ $0 \neq |\bar{M}_\epsilon \bar{A}| = |\bar{A}\bar{M}_\epsilon|$

$$= 3\epsilon \begin{vmatrix} 2-3\epsilon & 0 \\ 1-3\epsilon & 1 \end{vmatrix}$$

$$= 3\epsilon(2-3\epsilon)$$

i.e if and only if

$$\underline{\underline{\epsilon \notin \{0, 2/3\}}}$$

1 (b) $\bar{M}_t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ solves if

$$\underbrace{\bar{A}\,\bar{M}_t}_{=\bar{P}_t} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Now $\bar{P}_t = \bar{\bar{I}}$ if $\underline{\underline{t = 1/3}}$

1 (c) (i) Since $\bar{P}_{1/3} = \bar{\bar{I}}$,

$$\bar{A}\,\bar{M}_{1/3} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \forall\, a, b, c.$$

(ii) Because $|\bar{A}\,\bar{M}_{1/3}| = |\bar{\bar{I}}| = 1$, $\bar{A}^{-1}$ exists and there <u>cannot be other solutions</u>.

2 (a)

$$\int_1^T t\, e^{4t}\, dt = \Big|_1^T \left(t \cdot \tfrac{1}{4} e^{4t}\right) - \tfrac{1}{4}\int_1^T e^{4t}\, dt$$

$\uparrow$ u $\uparrow$ v'

$$= \frac{T}{4} e^{4T} - \frac{e^4}{4} - \frac{1}{16}\left(e^{4T} - e^4\right)$$

$$= \frac{1}{16}\left(e^{4T}[4T-1] - 3e^4\right)$$

2 (b)

Constant solution $x \equiv 0$.

Otherwise:

$$x^{-2}\,dx = 3\,t\,e^{4t}\,dt$$

→ note

$$-x^{-1} = 3\cdot\frac{e^{4t}}{16}(4t-1) - C$$

(from (a))

so

$$x(t) = \frac{1}{C + 3\cdot\frac{1-4t}{16}\,e^{4t}}$$

or $x(t) \equiv 0$.

You do not need to write this:

The note:

→ We need the general antiderivative of $t e^{4t}$. From (a) we have one.

→ $\frac{3}{16}e^4$ is just another constant.

→ "$+C$" vs "$-C$": both $\pm C$ are arbitrary constants

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2 (c)

[I presume most of you would use a formula from the book; there are two, one for the general solution and one for the initial value problem. You can choose what suits you!]

$$\frac{d}{dt}\left(x e^{-3t}\right) = \dot{x} e^{-3t} - 3x e^{-3t} = (\dot{x} - 3x)e^{-3t}$$

$$= t e^{7t} e^{-3t} = t e^{4t}.$$

Integrate from 1 to T using (a) and $x(1)$:

$$x(T)e^{-3T} - x(1)e^{-3} = \int_1^T t e^{4t} dt$$

$$x(T) = e^{3T}\left[2e^{-3} + \frac{e^{4T}}{16}(4T-1) - \frac{3}{16}e^{4}\right]$$

$$\underline{\underline{x(t) = \frac{1}{16} e^{7t}(4t-1) + e^{3t-3}\left[2 - \frac{3e^{7}}{16}\right]}}$$

3 $f(x,y) = \ln x + xy - y^2 - 2y\sqrt{3}$

(a) $f'_x(x,y) = \frac{1}{x} + y$

$f'_y(x,y) = x - 2y - 2\sqrt{3}$

St. pt: $y = -1/x$ yields

$$0 = x + 2/x - 2\sqrt{3}$$

$$0 = x^2 - x \cdot 2\sqrt{3} + 2$$

$$x = \tfrac{1}{2}\left[2\sqrt{3} \pm \sqrt{12 - 8}\right]$$

$$x_1 = \sqrt{3} - 1 \qquad x_2 = \sqrt{3} + 1$$

$$y_1 = -\frac{1}{\sqrt{3}-1} \qquad y_2 = -\frac{1}{\sqrt{3}+1}$$

$$= -\frac{\sqrt{3}+1}{(\sqrt{3}-1)(\sqrt{3}+1)} \qquad = -\frac{\sqrt{3}-1}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$= -\frac{\sqrt{3}+1}{2} \qquad = -\frac{\sqrt{3}-1}{2}$$

Stationary points:

$$(x_1, y_1) = \underline{\left(\sqrt{3}-1, -\frac{\sqrt{3}+1}{2}\right)}$$

$$(x_2, y_2) = \underline{\left(\sqrt{3}+1, -\frac{\sqrt{3}-1}{2}\right)}$$

Classify: next.

3 (a) Cont'd: Classify.

$f''_{xx}(x,y) = -\frac{1}{x^2}$ (<0)

$f''_{xy}(x,y) = f''_{yx}(x,y) = 1$

$f''_{yy}(x,y) = -2$ (<0)

(Stationary points are not loc min!)

Hessian $= \frac{2}{x^2} - 1$ which is >0 (≥ 0) if and only if $|x| < \sqrt{2}$. $(\leq \sqrt{2})$

$x_1 = \sqrt{3} - 1 \in (0,1)$ so

(x_1, y_1) is <u>local max</u>

$x_2 = \sqrt{3} + 1 > \sqrt{2}$ so

(x_2, y_2) is a <u>saddle point</u>

3 a cont'd

3 (b) No local min $\Rightarrow$ no global min.

$f(x,0) \to +\infty$ as $x \to +\infty$

$\Rightarrow$ no global max

Another remark you do not have to write on the exam:

* To justify "no global max" it suffices to show that f could take arbitrarily large positive values. Letting x grow with $y=0$ works, and also with $y=4$ or $y=10\,000$. But not $f(x,-1)$ $\to$ that gives no information.

* Alternative argument: If there is global max it has to be at (x_1, y_1).
$f(x_1,y_1) = \ldots\ldots \leq \ln x_1 + 3+\sqrt{3} < 3+\sqrt{3}$
But $f(e^R,0) = \ln e^R = R$.
Choose $R \geq 3+\sqrt{3}$; then (x_1, y_1) is not global max.

3 (c) Let

$$L(x,y) = f(x,y) - \lambda(x+y-1) + \mu y$$

Conditions:

$$\boxed{\begin{array}{ll} 0 = \frac{1}{x} + y - \lambda & (1) \\ 0 = x - 2y - 2\sqrt{3} - \lambda + \mu & (2) \\ \lambda \geq 0 & (3a) \\ \text{and } \lambda = 0 \text{ if } x+y<1 & (3b) \\ \mu \geq 0 & (4a) \\ \text{and } \mu = 0 \text{ if } \mu > 0 & (4b) \end{array}}$$

For $(x,y) = (1,0)$

(1): $\lambda = 1$ OK

(2): $0 = 1 - 2\sqrt{3} - 1 + \mu$

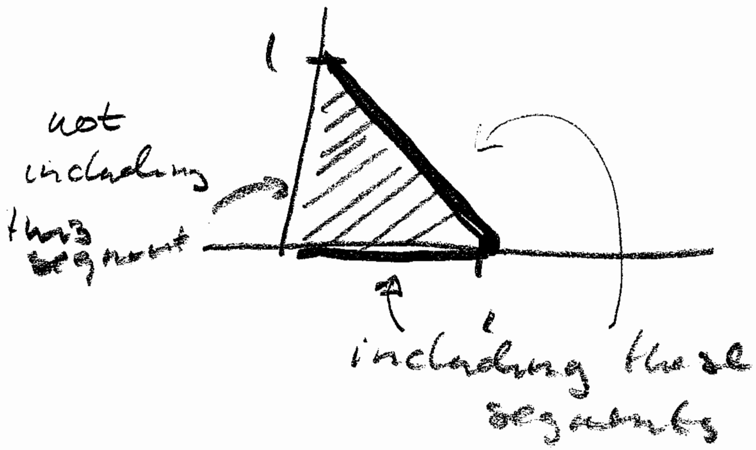
so $\mu = 2\sqrt{3}$ (≥ 0), OK

(3) OK $(x+y=1)$

(4) OK $(y=0)$

(1,0) satisfies the conditions

3 (d) The admissible set is



Constraints are linear so L and f have same Hessian. $\frac{2}{x^2} - 1$ which > 0 since $x \in (0, 1]$.

And $L''_{yy} = -2 < 0$

So L is concave and since (1,0) satisfies Kuhn-Tucker (and is admissible!) it solves the problem

3 e Tightening the $y \geq 0$ to $y \geq \frac{1}{200}$:

$$\text{Reduction} \approx \mu \cdot \frac{1}{200} \quad , \mu = 2\sqrt{3}$$

$$= \frac{\sqrt{3}}{100}$$

Problem 4

A remark first:
We did not even think of the function $h \equiv 0$, which is homogeneous of all degrees k
($\Rightarrow H = k$ which is homogeneous of degree 0.)
I hope the exam committee did not bother.

In the following, the $h \equiv 0$ function is disregarded.

Homotheticity: H = a homogeneous + a constant and <u>is homothetic</u> no <u>matter what k</u>.

Problem 4: When is H homogeneous?

→ if there is a q such that

$$H(tx, ty) = t^q H(x,y)$$

$$\underbrace{h(tx,ty)}_{t^k h} + k = t^q h(x,y) + t^q k$$

$$\left(t^k - t^q\right) h(x,y) = \left(t^q - 1\right) k \qquad (*)$$

valid for all (x,y), all $t>0$.

The RHS depends not on (x,y), so the LHS cannot either.

$\Downarrow$

$k = q$ or $h \equiv \bar{h}$ (constant)

$\Downarrow$ $(*)$ reads $0 = (t^k - 1)k$ $\Downarrow$ $\underline{k=0}$ for which H is homogeneous.

$\Downarrow$ $H(x,y) = \bar{h} + k$ which $\underline{\text{is}}$ homogeneous $(q=0)$. The case $h \equiv \bar{h}$ means that h is homogeneous of degree $\underline{k=0}$

In either case, $\underline{H \text{ is homogeneous}}$ $\underline{\text{when } h \text{ is homogeneous of degree } 0}$ (and only then).

[In fact this covers the $h \equiv 0$ too] 4#2