ECON3120/4120 Mathematics 2

December 8th 2014, 1430–1730.

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as pocket calculators.

Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Problem 1 Consider for each real number t the matrix $\mathbf{A}_t$ and the equation system (in the unknown (x, y, z)) given as follows:

$$\mathbf{A}_t \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \end{pmatrix} \qquad \text{where} \quad \mathbf{A}_t = \begin{pmatrix} 7 & 4 & 0 \\ 3t & -8 & t-7 \\ -6 & 3t & 2t+3 \end{pmatrix}$$

(a) Find $r \neq 0$ and $s \neq 0$ such that the determinant of $\mathbf{A}_t$ equals $t \cdot (rt + s)$.

(b) Except for two values t_0 and t_1 for t, the equation system has one and only one solution. Find t_0 and t_1.

(c) There are infinitely many solutions for precisely *one* of the t_0, t_1.
Solve the system for that t. (Do not do anything about the other t-value.)
Hint: From the previous parts it should be easy to spot which t.

Problem 2

(a) Use integration by substitution to show that $\displaystyle\int \frac{1}{x \ln|x|}\, dx = \ln\big|\ln|x|\big| + C$.
(Integration by substitution is mandatory. There is no score for differentiating the right-hand side.)

(b) Find the general solution of the differential equation

$$\dot{x} = (x \ln x)(1 + \ln t), \qquad t \geq 1,\ x \geq 1 \tag{D}$$

(c) Find the particular solution which passes through the point $(t, x) = (1, 1)$.

Problem 3 Let $f(x, y) = e^{1-x^3-y^4} - 1$.

(a) i) Find real numbers p and q such that the function $M(x, y) = f(x, y) - px - qy$ has a stationary point at $(x, y) = (1, 0)$.

ii) Classify $(x, y) = (1, 0)$ as a stationary point for M. (You can do this without having found p and q.)

Consider from now on the problem

$$V = \max\ f(x, y) \qquad \text{subject to } (x, y) \in S,$$

$$\text{where } S \text{ is given by the constraints} \quad \begin{cases} y \geq 0 \\ 2y \leq x - 1 \\ x \leq 2014 \end{cases} \tag{P}$$

(b) Explain why the problem has a solution, and state the Kuhn–Tucker conditions associated with the problem.

(c) Let (x, y) satisfy the Kuhn–Tucker conditions and the constraints stated in (P). Show that we must have $2y = x - 1$. (*Hint:* Suppose for contradiction that $2y \neq x - 1$.)

The point $(x, y) = (1, 0)$ solves the problem (P) (you shall not show this). If we replace the constraint « $y \geq 0$ » by « $y \geq -0.02$ », the optimal value increases by ΔV.

(d) Approximate ΔV from the Kuhn–Tucker conditions for (P). (You are asked for the approximation, not for the exact value.)

Problem 4 Define a function $H = H(x_1, \ldots, x_n)$ by

$$H(x_1, \ldots, x_n) = \left[x_1^{2014} + \ldots + x_n^{2014}\right]^{1/2014}$$

Without calculating derivatives or elasticities, find (for $H \neq 0$)

$$\mathrm{El}_1 H(x_1, \ldots, x_n) + \ldots + \mathrm{El}_n H(x_1, \ldots, x_n)$$

where $\mathrm{El}_i H$ denotes the partial elasticity $\dfrac{x_i}{H} \cdot \dfrac{\partial H}{\partial x_i}$.
(*Hint:* Calculate $H(tx_1, \ldots, tx_n)$; what is known about such functions?)