University of Oslo / Department of Economics / NCF

ECON3120/4120 Mathematics 2 – on the 2014–12–08 exam

- This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
  - This version reflects what was expected in this particular semester, and which may not be applicable to future semesters. In particular, what tests one is required to perform before answering «no conclusion» may not apply for later.
- For readability, the problems are restated, their respective solutions on the same page.
- Weighting: assigned at the grading committee's discretion. (In case of appeals: the new grading committee assigns weighting at their discretion.) The problem set was written with the intention that a uniform weighting over letter-enumerated items should be a *feasible* choice, and this – along with it being merely an *intention to facilitate* which does not tie the committe's hands – has been communicated.

Problem 4 fits the rest of this page:

Problem 4 Define a function $H = H(x_1, \ldots, x_n)$ by

$$H(x_1, \ldots, x_n) = \left[x_1^{2014} + \ldots + x_n^{2014}\right]^{1/2014}$$

Without calculating derivatives or elasticities, find (for $H \neq 0$)

$$\mathrm{El}_1 H(x_1, \ldots, x_n) + \ldots + \mathrm{El}_n H(x_1, \ldots, x_n)$$

where $\mathrm{El}_i H$ denotes the partial elasticity $\dfrac{x_i}{H} \cdot \dfrac{\partial H}{\partial x_i}$.
(*Hint:* Calculate $H(tx_1, \ldots, tx_n)$; what is known about such functions?)

On the solution: $H(t\mathbf{x}) = tH(\mathbf{x})$ so the function is homogeneous, and from Euler's homogeneous function theorem the answer is the degree of homogeneity, namely $\underline{\underline{1}}$.

Note: It was intentional to test whether the students recognize a homogeneous function from the defining property, omitting the «homogeneous» word and forbidding derivatives/elasticities calculated out. The elasticities formulation of Euler's theorem was lectured.

Problem 1 Consider for each real number t the matrix $\mathbf{A}_t$ and the equation system (in the unknown (x, y, z)) given as follows:

$$\mathbf{A}_t \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t \\ t \\ t \end{pmatrix} \qquad \text{where} \quad \mathbf{A}_t = \begin{pmatrix} 7 & 4 & 0 \\ 3t & -8 & t-7 \\ -6 & 3t & 2t+3 \end{pmatrix}$$

(a) Find $r \neq 0$ and $s \neq 0$ such that the determinant of $\mathbf{A}_t$ equals $t \cdot (rt + s)$.

(b) Except for two values t_0 and t_1 for t, the equation system has one and only one solution. Find t_0 and t_1.

(c) There are infinitely many solutions for precisely *one* of the t_0, t_1.
Solve the system for that t. (Do not do anything about the other t-value.)
Hint: From the previous parts it should be easy to spot which t.

On the solution:

(a) Cofactor expansion along the first row yields

$$\begin{aligned} |\mathbf{A}_t| &= 7 \begin{vmatrix} -8 & t-7 \\ 3t & 2t-3 \end{vmatrix} - 4 \begin{vmatrix} 3t & t-7 \\ -6 & 2t-3 \end{vmatrix} \\ &= -56(2t-3) - 21t(t-7) - 12t(2t-3) - 24(t-7) \\ &= t^2 \cdot (-21 - 24) + t \cdot (-108 + 147 - 24) \end{aligned}$$

so that $\underline{\underline{r = -45}}$ and $\underline{\underline{s = -25}}$.

(b) Unique solution if and only if $|\mathbf{A}_t| \neq 0$; from the information given in part (a), $\underline{\underline{t_0 = 0}}$ and $t_1 = -s/r$. The only thing where the *calculations* from (a) are needed, is to insert s and r to obtain $\underline{\underline{t_1 = -5/9}}$.

(c) From the information given in part (a), there are infinitely many solutions when $t = 0$ (since then $|\mathbf{A}_t| = 0$ and the system is homogeneous). We have

$$\mathbf{A}_0 = \begin{pmatrix} 7 & 4 & 0 \\ 0 & -8 & -7 \\ -6 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 0 & 4 & 7/2 \\ 0 & -8 & -7 \\ -1 & 0 & 1/2 \end{pmatrix}$$

by scaling the third row by $1/6$ and adding 7 of it to the first. The two first rows are now proportional. Now let $\underline{\underline{x = c}}$, so that $\underline{\underline{z = 2c}}$ and $\underline{\underline{y = -7c/4}}$, ($c$ being free).

(Other solution formulations are of course possible.)

Problem 2

(a) Use integration by substitution to show that $\displaystyle\int \frac{1}{x\ln|x|}\,dx = \ln\big|\ln|x|\big| + C$.
(Integration by substitution is mandatory. There is no score for differentiating the right-hand side.)

(b) Find the general solution of the differential equation

$$\dot{x} = (x\ln x)(1+\ln t), \qquad t \geq 1,\ x \geq 1 \tag{D}$$

(c) Find the particular solution which passes through the point $(t, x) = (1, 1)$.

On the solution:

(a) Both $u = \ln|x|$ and $v = \ln\big|\ln|x|\big|$ work, the former is likely more intuitive; $du = dx/x$, transforming the integral to

$$\int \frac{du}{u} = \ln|u| + C = \ln\big|\ln|x|\big| + C$$

(b, c) Since $x \geq 1$, we have only one constant solution, namely $\underline{\underline{x \equiv 1}}$ (this answers (c)). For $x > 1$, we separate and integrate

$$\int \frac{dx}{x\ln x} = \int (1+\ln t)\,dt = K + t\ln t$$

where for the RHS integral we have used the information given in part (a) (since $x > 1$, no absolute value signs), and the dt-integral is solved by parts $\int 1\cdot\ln t\,dt = t\ln t - \int \frac{t}{t}dt = t\ln t - t + K$. So with Q being an arbitrary nonnegative constant ($Q = 0$ for the constant solution, otherwise $Q = e^K$), the answer to part (b) is

$$\underline{\underline{x(t) = e^{Qt^t}, \qquad (Q \geq 0)}}$$

Problem 3 Let $f(x,y) = e^{1-x^3-y^4} - 1$.

(a) i) Find real numbers p and q such that the function $M(x,y) = f(x,y) - px - qy$ has a stationary point at $(x,y) = (1,0)$.

ii) Classify $(x,y) = (1,0)$ as a stationary point for M. (You can do this without having found p and q.)

On the solution of part (a): For grading, one has to clarify the level of ambition as has been announced for this semester: Local second-order conditions need only be checked *at* the point, and will be inconclusive if the Hessian determinant vanishes there – then «inconclusive» is a perfectly valid answer. They may of course employ the punctured neighbourhood version, but it is not required in *this* semester. (This cannot be taken to be valid in semesters to follow.)

Thus from the partial first derivatives

$$M'_x(x,y) = -3x^2 e^{1-x^3-y^4} - p \qquad \text{and} \qquad M'_y(x,y) = -4y^3 e^{1-x^3-y^4} - q,$$

we have $\underline{\underline{p = -3}}$ and $\underline{\underline{q = 0}}$ for $(1,0)$ to be stationary, and from the second derivatives

$$\begin{aligned}
M''_{xx}(x,y) &= f''_{xx}(x,y) = \left[(-3x^2)^2 - 6x\right] e^{1-x^3-y^4} && \text{so that} \quad M''_{xx}(1,0) = 3 \\
M''_{yy}(x,y) &= f''_{yy}(x,y) = \left[(-4y^3)^2 - 12y^2\right] e^{1-x^3-y^4} && \text{so that} \quad M''_{yy}(1,0) = 0 \\
M''_{xy}(x,y) &= f''_{xy}(x,y) = -12x^2y^3 e^{1-x^3-y^4} && \text{so that} \quad M''_{xy}(1,0) = 0 = M''_{yx}(1,0)
\end{aligned}$$

the second-derivative test $\underline{\underline{\text{is inconclusive}}}$.

It is of course perfectly OK to make e.g. the following argument why we have a saddle point: Because $M''_{xx}(1,0) = 3$, then for fixed $y = 0$ we have a local min for $M(x,0)$; because M''_{yy} is $4y^2[4y^4 - 3]e^{1-x^3-y^4}$ which is negative for $y \neq 0$, then for fixed $x = 1$ we have a local max for $M(1,y)$. Thus $(1,0)$ is neither local min nor local max, i.e. it is a saddle point.

[problem 3 cont'd:] Consider from now on the problem

$$V = \max\ f(x,y) \qquad \text{subject to } (x,y) \in S,$$

$$\text{where } S \text{ is given by the constraints} \quad \begin{cases} y \geq 0 \\ 2y \leq x - 1 \\ x \leq 2014 \end{cases} \tag{P}$$

(b) Explain why the problem has a solution, and state the Kuhn–Tucker conditions associated with the problem.

(c) Let (x, y) satisfy the Kuhn–Tucker conditions and the constraints stated in (P). Show that we must have $2y = x-1$. (*Hint:* Suppose for contradiction that $2y \neq x-1$.)

The point $(x, y) = (1, 0)$ solves the problem (P) (you shall not show this). If we replace the constraint « $y \geq 0$ » by « $y \geq -0.02$ », the optimal value increases by ΔV.

(d) Approximate ΔV from the Kuhn–Tucker conditions for (P). (You are asked for the approximation, not for the exact value.)

On the solution, part (b) ff.:

(b) Existence by the extreme value theorem, as f is continuous and S is closed and bounded (and nonempty).
Putting $L(x, y) = f(x, y) - \lambda(2y - x + 1) + \alpha y - \beta(x - 2014)$, the conditions are:

$$\begin{aligned}
&0 = -3x^2 e^{1-x^3-y^4} + \lambda - \beta \\
&0 = -4y^3 e^{1-x^3-y^4} - 2\lambda + \alpha \\
&\lambda \geq 0 \qquad (= 0 \text{ if } 2y - x < -1) \\
&\alpha \geq 0 \qquad (= 0 \text{ if } y > 0) \\
&\beta \geq 0 \qquad (= 0 \text{ if } x < 2014)
\end{aligned}$$

Note: Conventions vary over what condition set bears the «Kuhn–Tucker» name;

- It is OK to include or to omit the admissibility conditions.
- Variants that are logically equivalent – e.g. with «$\alpha \geq 0 = \alpha y$» are also just fine. (The above does not write «$-\alpha(-y)$» either.)

(c) Suppose for contradiction that the Kuhn–Tucker conditions hold with $2y < x - 1$. Then $\lambda = 0$ and since $x \neq 0$ on S, we have the contradiction $\beta < 0$.

(d) The increase is 0.02 times the appropriate multiplier, which in the above notation is α. Since $\beta = 0$ we have $\lambda = 3$ and since $y = 0$ we have $\alpha = 2\lambda = 6$. So the answer is $\underline{\underline{0.12}}$.