University of Oslo / Department of Economics **(English version.)**

ECON3120/4120 Mathematics 2

May 29th 2015, 0900–1200.

There are 2 pages of problems to be solved.

All printed and written material may be used, as well as pocket calculators.

Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier part or item (e.g. "(a)" or "i)") to solve a later one (e.g. "(c)" or "ii)"), regardless of whether you managed to answer the former. A later part or item does not necessarily utilize answers from or information given in a previous one.

Problem 1 Define for each real number w the matrix $\mathbf{A}_w$ and the vector $\mathbf{b}$ by

$$\mathbf{A}_w = \begin{pmatrix} 1000+w & 1001 & 1002 \\ 1000 & 1000+w & 1000 \\ 1000 & 999 & 998+w \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

(a) i) Calculate $\mathbf{A}_2\mathbf{b}$.

ii) Why does it follow from i) that the determinant $|\mathbf{A}_2|$ equals 0?

(b) Observe that the sum of the three rows is $(3000+w)$ times the vector $(1,1,1)$. Use this to show that i) $|\mathbf{A}_{-3000}| = 0$, and ii) $|\mathbf{A}_0| = 0$.

(c) For each of the following equation systems, decide the number of solutions (zero, one or more than one): i) $\mathbf{A}_0\mathbf{x} = \mathbf{0}$; ii) $\mathbf{A}_1\mathbf{x} = \mathbf{1}$; iii) $\mathbf{A}_2\mathbf{x} = \mathbf{b}$.
(*Hint:* It may help to add two of the equations to the third.)

Problem 2 The following equation system defines continuously differentiable functions $u = u(p,q)$, $v = v(p,q)$ around the point where $p = q = 1$, $u = v = 0$:

$$\begin{aligned} e^u + \ln(1+pqv) + uv - q &= 0 \\ e^u - e^{uv} - p &= -1 \end{aligned} \tag{E}$$

(a) Differentiate the system (i.e. calculate differentials).

(b) Find a general expression for $v'_p(p,q)$.

Problem 3

(a) Find the limits (or show non-existence), for arbitrary constants $p > q \geq 0$. (*Hint for* $p \in (0,1)$*: Write* $\frac{1}{t} = \frac{t-1}{t} \cdot \frac{1}{t-1}$.)

$$\text{i)} \quad \lim_{t\to 1} \frac{(\ln t)^{q+1}}{t-1}, \qquad \text{ii)} \quad \lim_{t\to +\infty} \frac{\ln t}{(t-1)^p}, \qquad \text{iii)} \quad \lim_{t\to +\infty} \frac{(\ln t^p)^{q+1}}{(t-1)^p}$$

For each $x_1 > 0$, let $x(t)$ be that particular solution of the differential equation

$$\dot{x} = \frac{x^3 - 8}{x^2} \cdot t^8 \ln t \tag{D}$$

which is such that $x(1) = x_1$. (The equation is valid only for $t > 0$, $x > 0$.)

(b) Explain why $\ddot{x}(1) = (x_1 - \dfrac{8}{x_1^2}) \cdot \lim\limits_{t\to 1} \dfrac{\ln t}{t-1}$, and use this and part (a) to find the quadratic approximation of x around $t = 1$.
For full score, you are required to show it using only these pieces of information. If you use other means, e.g. part (c) below, you can still get up to a "C" worth of score.

(c) Solve (D) for each value of $x_1 > 0$.

Problem 4 Let $p \neq q$ be positive integers with q odd (i.e. 1, 3, 5, ...). Let $g(x, y, z) = x^{q+1} + y^{q+1} + z^{q+1} - 1$, and consider the max/min problems

$$\max / \min \ \frac{x^{p+1} + y^{p+1} + z^{p+1}}{p+1} \qquad \text{subject to } g(x,y,z) = 0 \qquad \text{and} \quad x + y - z = 1 \quad \text{(P)}$$

(a) i) At least one of the max/min problems will have a solution. Which one(s)? (*Hint:* It is crucial that q is odd so that i.e. $g(-x, y, z) = g(x, y, z)$.)

ii) State the associated Lagrange conditions.

In the rest of this problem, we shall consider possible solutions of the form $(x, y, z) = (x, x, 2x - 1)$. Set $y = x$, and $z = x + y - 1 = 2x - 1$, so that the first constraint reads $h(x) = 0$, where $h(x) = g(x, x, 2x - 1) = 2x^{q+1} + (2x - 1)^{q+1} - 1$. (You shall not show this.)

(b) i) Show that the function $h(x)$ has a zero $x^* > 1/2$.

ii) Show that the point $(x, y, z) = (x^*, x^*, 2x^* - 1)$ satisfies the Lagrange conditions, where $x^* > 1/2$ is the zero from item i).

- *If unable to do so,* score up to a "C" on this item ii) may be awarded if you instead show that the Lagrange conditions are satisfied at $(0, 0, -1)$.