University of Oslo / Department of Economics / NCF

ECON3120/4120 Mathematics 2 – on the 2015–05–29 exam

- This note is *not* suited as a complete solution or as a template for an exam paper, it is too sketchy. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
- For readability, the problems are restated, followed by their respective solutions.
- Weighting: assigned at the grading committee's discretion. (In case of appeals: the new grading committee assigns weighting at their discretion.) The problem set was written with the intention that a uniform weighting over letter-enumerated items should be a *feasible* choice, and this – along with it being merely an *intention to facilitate* which does not tie the committe's hands – has been communicated.

Questions that took more exam time than usual The grading committee needs to be aware that there were a lot of questions to this exam problem set. I think there was indeed more (legitimate!) need for clarification than usual, and the time it took to walk around and respond to questions may impair the efficiency also to the majority that did not have any questions.
The time spent amounted to approximately 40 minutes, from 1005 to 1045 (and I was subsequently called back at 11 for a question from a single candidate). Two plenary announcements were made, and the conditions in *Idrettsbygningen* meant several candidates needed to have the questions repeated individually. Arguably, this left the candidates with less than three effective hours on this problem set.
The following summarizes the issues (not counting those which only one candidate had questions on). Some of these are issues that the candidates would under normal circumstances have been expected to figure out by themselves, but which were clarified in the interest of time, and also because they concerned intended *hints,* that likely did not serve their purpose to everyone.

- 1 (b). The sum of the rows means the sum of row vector number one, row vector number two, and row vector number 3. (And the (usual) questions of whether the notation "$\mathbf{A}_{-3000}$" really means to put $w = -3000$. Affirmed to those who asked.)
- 1 (c), *plenary announcement due to several questions on the matter:*
  The hint could have been taken to mean to add up the three vector equations i), ii) and iii). It was clarified that the hint concerns each of them. Most likely the problem set would have done better without this hint, which was intended to facilitate recycling of the calculations from part (b).
- 2, *plenary announcement:*
  What general expression means, and that $p = q = 1$, $u = v = 0$ merely identifies the function. They should know, but the combination of pitfalls and *zeitnot* ...

- 3 (a), *plenary announcement:*
  The "p" and "q" letters used in problems 2, 3(a), 4 shall not be taken to mean the problems are related – in particular, the ranges could be different, and in particular they should in part 3 (a) take care to cover the possible case $q = 0$.
  The latter is something they would normally be expected to catch, but there is a risk that someone would just assume q being natural (cf. problem 4) – even if the hint is stated to be useful for a non-integer range for p.

- 4, *plenary announcement:*
  There is *one* max.-problem and *one* min.-problem, and both constraints apply to both. It is not to be taken as one (or two?) problem(s) for each constraint. Notice the (unintended) slight difference in wording between Norwegian and English version.

The committee should apply their best judgment on the impact of the issues.

Problem 1 Define for each real number w the matrix $\mathbf{A}_w$ and the vector $\mathbf{b}$ by

$$\mathbf{A}_w = \begin{pmatrix} 1000+w & 1001 & 1002 \\ 1000 & 1000+w & 1000 \\ 1000 & 999 & 998+w \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

(a) i) Calculate $\mathbf{A}_2\mathbf{b}$.

ii) Why does it follow from i) that the determinant $|\mathbf{A}_2|$ equals 0?

(b) Observe that the sum of the three rows is $(3000 + w)$ times the vector $(1, 1, 1)$. Use this to show that i) $|\mathbf{A}_{-3000}| = 0$, and ii) $|\mathbf{A}_0| = 0$.

(c) For each of the following equation systems, decide the number of solutions (zero, one or more than one): i) $\mathbf{A}_0\mathbf{x} = \mathbf{0}$; ii) $\mathbf{A}_1\mathbf{x} = \mathbf{1}$; iii) $\mathbf{A}_2\mathbf{x} = \mathbf{b}$.
(*Hint:* It may help to add two of the equations to the third.)

On the solution of problem 1 This problem tests calculations of determinants – this specifically through row operations; furthermore, the connection between nonzero determinant and uniqueness of solution; and – in (c) item iii) – Gaussian elimination.

(a) i) $w = 2$ yields $\mathbf{A}_2\mathbf{b} = (1002 - 1002,\ 1000 - 1000,\ 1000 - 1000)' = \mathbf{0}$.

ii) That means $|\mathbf{A}_2| = 0$; had it been nonzero, then $\mathbf{A}_2\mathbf{x}$ could only be $\mathbf{0}$ for $\mathbf{x} = \mathbf{0}$ (we would have had $\mathbf{b} = \mathbf{A}_2^{-1}\mathbf{0}$).

(b) Adding row #1 and row #2 to row #3 does not change the determinant, so that

$$|\mathbf{A}_w| = \begin{vmatrix} 1000+w & 1001 & 1002 \\ 1000 & 1000+w & 1000 \\ 3000+w & 3000+w & 3000+w \end{vmatrix} = (3000+w)\begin{vmatrix} 1000+w & 1001 & 1002 \\ 1000 & 1000+w & 1000 \\ 1 & 1 & 1 \end{vmatrix}$$

where we have used that a scaling of a single row (the common $3000+w$ in the third row) will factor out. This is zero for $w = -3000$. For $w = 0$, we have two proportional rows and thus zero determinant.

(c) From (b) we it is given that $|\mathbf{A}_w|$ vanishes when $w \in \{-3000, 0\}$. The calculations from (a) imply $|\mathbf{A}_2| = 0$.

i) Homogeneous system, singular coefficient matrix: infinitely many solutions.

ii) There are basically three ways to show that the coefficient matrix is invertible, thus there is a unique solution.

- One is to calculate the determinant completely. Subtracting $1000+w$ resp. 1000 of the third row from the others, yields

$$|\mathbf{A}_w| = (3000+w)\begin{vmatrix} 1000+w & 1001 & 1002 \\ 1000 & 1000+w & 1000 \\ 1 & 1 & 1 \end{vmatrix}$$
$$= (3000+w)\begin{vmatrix} 0 & 1-w & 2-w \\ 0 & w & 0 \\ 1 & 1 & 1 \end{vmatrix} = (3000+w)w(w-2)$$

  so that $|\mathbf{A}_1| \neq 0$ and there is unique solution.

- Alternatively – shorter, but a bit "fancy" – is to notice that the determinant involves one cubic term and none of higher order, so there cannot be more than three zeroes – and we have found three, and 1 is not among them. Thus, again, $|\mathbf{A}_1| \neq 0$ and unique solution.

- Or one could start eliminating, to sooner or later arrive at either the solution itself, or at an invertible coefficient matrix.

iii) Put $w = 2$ and solve. The hint gives nice numbers: summing the equations – say, adding the first and second to the third – yields $3002(x+y+z) = 0$, and dividing by 3002 the third equation reads $x+y+z = 0$. Subtracting 1000 of this from #2 yields $2x_2 = 0$, subtracting 1002 of it from #1 yields $-x_2 = 1$, so we have no solution.

Eliminating in a different order is of course perfectly fine.

Problem 2 The following equation system defines continuously differentiable functions $u = u(p,q)$, $v = v(p,q)$ around the point where $p = q = 1$, $u = v = 0$:

$$\begin{aligned} e^u + \ln(1+pqv) + uv - q &= 0 \\ e^u - e^{uv} - p &= -1 \end{aligned} \tag{E}$$

(a) Differentiate the system (i.e. calculate differentials).

(b) Find a general expression for $v_p'(p,q)$.

On the solution of problem 2 This problem tries the familiarity with differentials, and the ability to identify a *linear* equation system for dv (and solve it).

Term by term differentiation:

$$\begin{aligned} e^u\,du + \frac{qv\,dp + pv\,dq + pq\,dv}{1+pqv} + v\,du + u\,dv - dq &= 0 \\ e^u\,du - e^{uv}(v\,du + u\,dv) - dp &= 0 \end{aligned}$$

or, collecting terms:

$$\begin{aligned} \big[e^u + v\big]\,du + \big[u + \frac{pq}{1+pqv}\big]\,dv &= -\frac{qv}{1+pqv}\,dp + \big[1 - \frac{pv}{1+pqv}\big]\,dq \\ \big[e^u - ve^{uv}\big]\,du - ue^{uv}\,dv &= dp \end{aligned}$$

For v_p' we want to solve for dv with $dq = 0$. For example we can use Cramér's rule:

$$v_p'(p,q) = \frac{\begin{vmatrix} e^u + v & \frac{-qv}{1+pqv} \\ e^u - ve^{uv} & 1 \end{vmatrix}}{\begin{vmatrix} e^u + v & u + \frac{pq}{1+pqv} \\ e^u - ve^{uv} & -ue^{uv} \end{vmatrix}} = -\frac{(1+pqv)(e^u+v) + qv(e^u - ve^{uv})}{(1+pqv)(e^u+v)ue^{uv} + ((1+pqv)u + pq)(e^u - ve^{uv})}$$

(after having expanded the fraction by $1+pqv$, which isn't essential). Note the minus sign. As mentioned, it was clarified that the numbers for p, q, u and v were only to identify the function.

Problem 3, part (a) first

(a) Find the limits (or show non-existence), for arbitrary constants $p > q \geq 0$. (*Hint for* $p \in (0,1)$*:* Write $\frac{1}{t} = \frac{t-1}{t} \cdot \frac{1}{t-1}$.)

$$\text{i)} \quad \lim_{t\to 1} \frac{(\ln t)^{q+1}}{t-1}, \qquad \text{ii)} \quad \lim_{t\to +\infty} \frac{\ln t}{(t-1)^p}, \qquad \text{iii)} \quad \lim_{t\to +\infty} \frac{(\ln t^p)^{q+1}}{(t-1)^p}$$

On the solution of problem 3 part (a) This part tests l'Hôpital (it was pointed out that they needed to cover $q = 0$, which gives a different-looking answer in item i)) and manipulating limits; the latter trick of taking the limit inside the power was given in a lecture.

(a) i) A "0/0" expression, l'Hôpital's rule yields $\lim_{t\to 1}(q+1)(\ln t)^q t^{-1} = 0$ for $q > 0$, while for $q = 0$ we get $\lim_{t\to 1} t^{-1} = 1$.

ii) An "∞/∞" expression, l'Hôpital's rule yields

$$\lim_{t\to\infty} \frac{1}{pt(t-1)^{p-1}} = \frac{1}{p} \lim_{t\to\infty} \left(\frac{t-1}{t} \cdot (t-1)^{-p}\right) = 0$$

for all $p > 0$, where we have used $(t-1)/t \to 1$.

iii) We have $\dfrac{(\ln t^p)^{q+1}}{(t-1)^p} = \left[\dfrac{p \ln t}{(t-1)^{p/(q+1)}}\right]^{q+1}$, and $(p \ln t)/(t-1)^{p/(q+1)}$ tends to zero by item ii) (because $p/(q+1) > 0$ – that need not be pointed out).

Problem 3, cont'd For each $x_1 > 0$, let $x(t)$ be that particular solution of the differential equation

$$\dot{x} = \frac{x^3 - 8}{x^2} \cdot t^8 \ln t \tag{D}$$

which is such that $x(1) = x_1$. (The equation is valid only for $t > 0$, $x > 0$.)

(b) Explain why $\ddot{x}(1) = (x_1 - \frac{8}{x_1^2}) \cdot \lim_{t \to 1} \frac{\ln t}{t - 1}$, and use this and part (a) to find the quadratic approximation of x around $t = 1$.
For full score, you are required to show it using only these pieces of information. If you use other means, e.g. part (c) below, you can still get up to a "C" worth of score.

(c) Solve (D) for each value of $x_1 > 0$.

On the solution of problem 3 (b) and (c) A bit akin to problem 127 (b) from the compendium, although with a quadratic approximation:

(b) The last part first: we have $x(t) \approx x_1 + \dot{x}(1) \cdot (t - 1) + \frac{1}{2}\ddot{x}(1) \cdot (t - 1)^2$. Because $\dot{x}(1) = (x_1^3 - 8x_1^{-2}) \cdot 1^8 \ln 1 = 0$, and the expression given for $\ddot{x}(1)$ yields $(x_1 - 8/x_1^2) \cdot 1$ by (a) item i) (with $q = 0$), the second-order approximation is

$$x(t) \approx x_1 + (\frac{x_1}{2} - \frac{4}{x_1^2}) \cdot (t - 1)^2$$

To arrive at the form for $\ddot{x}(1)$:

- The quick way: since $\dot{x}(1) = 0$, then (from the very definition of the derivative at 1 of $\dot{x}$, although one has to accept a reference to l'Hôpital's rule) we have $\ddot{x}(1) = \lim_{t\to 1} \frac{\dot{x}(t)}{t-1}$. Insert for $\dot{x}$ from (D). The "0/0"-expression $\ln t/(t - 1) \to 1$ converges, from (a) item i), and for the rest just insert 1 for t and x_1 for $x(1)$.
- The following is also perfectly fine: differentiate the expression for $\dot{x}$ to get

  $$\ddot{x}(t) = \Big[\frac{x^3 - 8}{x^2} \cdot t^8\Big]' \ln t + \Big[\frac{x^3 - 8}{x^2} \cdot t^8\Big] \frac{1}{t}.$$

  At $t = 1$ the first term vanishes (because of the ln), leaving us with $\ddot{x}(1) = x_1 - 8x_1^{-2}$. One then merely has to point out that $\frac{\ln t}{t-1} \to 1$.

Remarks to part (b): *The reason to restrict the available information, is to test whether they understand a differential equation says about the derivative. The committee has some discretion on awarding partial score, as the "C" typically ranges 55 to 75 percent score, but the left end of that interval was not intended.*
(If the candidates miss out the different form of the limit for $q = 0$, they could easily get an erroneous zero for the second derivative. If it is clear that they have the correct formula for the quadratic approximation – a "$\frac{1}{2}$" in particular – that error does hardly simplify away too much of the essence of the question.)

(c) For readability, the differential equation is restated on this page:

$$\dot{x} = \frac{x^3 - 8}{x^2} \cdot t^8 \ln t$$

We have a constant solution $x \equiv 2$ if $x_1 = 2$. For $x_1 \neq 2$, separate into $x^2 dx/(x^3 - 8) = t^8 \ln t \, dt$ and integrate. For the dx integral, substitute $y = x^3 - 8$, $dy = 3x^2 dx$, while integrate the dt integral by parts:

$$\begin{aligned} \frac{1}{3} \int \frac{dy}{y} &= \frac{1}{9} t^9 \ln t - \frac{1}{9} \int t^9 \frac{1}{t} \, dt, \quad \text{so that} \\ \frac{1}{3} \ln |y| &= \frac{1}{9} t^9 \ln t - \frac{1}{81} t^9 + C. \end{aligned}$$

The left-hand side is $\frac{1}{3} \ln |x^3 - 8|$. To get rid of the C, insert $t = 1$ and $x = x_1$ to get $C = \frac{1}{3} \ln |x_1^3 - 8| + \frac{1}{81}$. Insert, multiply by 3 and apply the exponential function:

$$|x(t)^3 - 8| = |x_1^3 - 8| \cdot e^{\frac{1}{3} \ln t \cdot t^9 - (t^9 - 1)/27}$$

Now $x(t) - 2$ will have the same sign as $x_1 - 2$ – start above (resp. below) the constant solution $\Rightarrow$ stay above (resp. below) it – so the absolute value signs can be dropped. Solve for x:

$$x(t) = \sqrt[3]{8 + (x_1^3 - 8) t^{(t^9)/3} \cdot e^{-(t^9 - 1)/27}}$$

(the parentheses around t^9 just for readability: it isn't the 9 that is divided). By inspection, this form also covers the constant solution.

Remarks to part (c): *Part (c) will most likely see quite a few simple mistakes in the calculations – those should not be penalized too harshly. One can also expect a few candidates to forget the absolute value signs, which is also a very minor point – it was not what was intended to test. The reason why the problem did not specify $x_1 > 2$, was in order to capture the constant solution; it has been stressed a bit in class, as it highlights the significance of not dividing by zero. So the constant solution is in my opinion way more significant than the absolute value.*

Problem 4 Let $p \neq q$ be positive integers with q odd (i.e. 1, 3, 5, ...). Let $g(x, y, z) = x^{q+1} + y^{q+1} + z^{q+1} - 1$, and consider the max/min problems

$$\max / \min \frac{x^{p+1} + y^{p+1} + z^{p+1}}{p+1} \qquad \text{subject to } g(x, y, z) = 0 \qquad \text{and} \quad x + y - z = 1 \quad \text{(P)}$$

(a) i) At least one of the max/min problems will have a solution. Which one(s)? (*Hint:* It is crucial that q is odd so that i.e. $g(-x, y, z) = g(x, y, z)$.)

ii) State the associated Lagrange conditions.

In the rest of this problem, we shall consider possible solutions of the form $(x, y, z) = (x, x, 2x - 1)$. Set $y = x$, and $z = x + y - 1 = 2x - 1$, so that the first constraint reads $h(x) = 0$, where $h(x) = g(x, x, 2x - 1) = 2x^{q+1} + (2x - 1)^{q+1} - 1$. (You shall not show this.)

(b) i) Show that the function $h(x)$ has a zero $x^* > 1/2$.

ii) Show that the point $(x, y, z) = (x^*, x^*, 2x^* - 1)$ satisfies the Lagrange conditions, where $x^* > 1/2$ is the zero from item i).

- *If unable to do so,* score up to a "C" on this item ii) may be awarded if you instead show that the Lagrange conditions are satisfied at $(0, 0, -1)$.

On the solution of problem 4 Here it was clarified that there is one min.-problem and one max.-problem – one can hope that nobody writes one max.-problem per constraint. This problem intends to test (a)i) the extreme value theorem (though not in full detail; this course sticks to functions that are continuous wherever defined, and though they are told that one must beware natural restrictions on the domain of definition, e.g. for logs, those concerns do not apply here; also, we do not stress non-emptiness of the admissible set, although $(0, 0, -1)$ is admissible); (a)ii) the Lagrange conditions; (b)i) the intermediate value theorem (b)ii) the ability to verify that there are multipliers that do the job for a given point. (Though, one cannot penalize those who deduce the point rather than taking it for granted and verifying.) It was deliberate to assign a *Lagrange* problem, so that one does not need to verify that there are *nonnegative* multipliers at the point(s).

(a) i) Both problems do have solution, by the extreme value theorem:
All functions are continuous (on the entire $\mathbf{R}^3$ space), and because q is odd, neither $|x|$ nor $|y|$ nor $|z|$ can exceed 1, so the set is bounded. It is also closed (no argument required) and nonempty (hardly necessary to even point out).

ii) With $L(x, y) = \dfrac{x^{p+1} + y^{p+1} + z^{p+1}}{p+1} - \lambda \cdot (x^{q+1} + y^{q+1} + z^{q+1} - 1) - \mu(x + y - z - 1)$, the Lagrange conditions can be written as e.g.

$$\begin{aligned} x^p &= (q+1)\lambda x^q + \mu \\ y^p &= (q+1)\lambda y^q + \mu \\ z^p &= (q+1)\lambda z^q - \mu \end{aligned} \qquad \begin{aligned} x^{q+1} + y^{q+1} + z^{q+1} &= 1 \\ x + y - z &= 1 \end{aligned}$$

(b) i) The intermediate value theorem yields a zero because $h(1/2) = 2 \cdot 2^{-(q+1)} - 1 = 2^{-q} - 1 < 0$ while $h(x) \to +\infty$ as $x \to +\infty$.

ii) It is given in the problem text that the constraints hold when x is a zero of h. When $x = y$ we have $L'_x = L'_y$, so the three stationarity conditions reduce to two:

$$(x^*)^p = (q+1)\lambda(x^*)^q + \mu \qquad \text{and} \qquad (2x^* - 1)^p = (q+1)\lambda(2x^* - 1)^q - \mu$$

It suffices to point out that this is a 2×2 linear equation system with nonsingular coefficient matrix, but if one tries to solve, adding the equations yield a λ-coefficient of $(q+1)\left[(x^*)^q + (2x^* - 1)^q\right]$ which is maybe so obviously positive that it will not even be pointed out. (On the other hand it is not obvious unless one has in mind that $x^* \geq 1/2$, but it is hard to argue ...) So there will be a λ, and thus a μ, such that the Lagrange conditions are satisfied.

- The alternative point $(0, 0, -1)$ is easier. The constraints easily hold (it is hardly required to point out that the odd q yields $(-1)^{q+1} - 1 = 0$), and inserting into the stationarity conditions we get two equations reading $\mu = 0$, while the third reads $(-1)^p = (q+1)\lambda(-1)^q$. Thus the conditions are satisfied for some λ and μ.