Linearization, differentials, eq. systems, implicit derivatives ...

Recall the logic underlying the MR(T)S formula $\frac{\partial F/\partial K}{\partial F/\partial L}$:

(That's good enough "expression for" ... that clarification will be summarized at the end of tomorrow.)

- Fixing a *level curve* (isoquant, indifference curve, ...) $F(K, L) = C$, will determine one variable in terms of the other(s). Say, $L = L(K)$.
- *Total* derivative wrt. K: $\frac{\partial F}{\partial K} + \frac{\partial F}{\partial L}\frac{\partial L}{\partial K} = 0$ (because C is constant, so $\frac{\partial C}{\partial K} = 0$). Solve out for $\frac{\partial L}{\partial K} = -\frac{\partial F}{\partial K}\Big/\frac{\partial F}{\partial L}$.

  (The MRS is the negative of this: how much must you increase one if you want to *reduce* the other.)

Topic for Tuesday (and likely a bit of Wednesday too):

- Suppose you have n equations determining ("endogenizing"?) n variables in terms of the other free (exogeneous, then?) variable(s). *What are the derivatives?*
  - Focus: $n = 2$. The "new stuff" arises as soon as $n > 1$.
  - Book: (u, v) as functions of (x, y) (or of (x, y, z)).
    But: you are expected to handle (x, y) as functions of (r, s, t) or (K, L) as functions of (p, w) or of $(p, w, a, b, \alpha, \beta, \gamma)$ or ...

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If the n equations were all *linear*, we would be able to solve (and know how!). But that's a very lucky case. Not so lucky example:

Production: sum of two Cobb–Douglases, $(1-\gamma)K^a L^b + \gamma K^\alpha L^\beta$.
Unit prices: p and w on K and L, respectively.
K and L determined by first-order cond'ns for profit maximization:

$$(1-\gamma)aK^{a-1}L^b + \gamma\alpha K^{\alpha-1}L^\beta = p$$
$$(1-\gamma)bK^a L^{b-1} + \gamma\beta K^\alpha L^{\beta-1} = w$$

Q: How do (K, L) depend on (p, w)? Or on everything else?

By Tuesday, we shall cover how to get *expressions for* the partial derivatives of K and L.

(Did you suggest to solve the FOC's for (K, L) explicitly, and then differentiate? Nah, only in very special cases you can. Since economics is so full of quantities *implicitly* given – for example in terms of FOC's – we need a way to handle derivatives of implicitly given functions.)

Differentials. ("Main tool" for avoiding too much new matrix-based terminology.)

- Input change from (K, L) to $(K + \Delta K, L + \Delta L) \rightsquigarrow$ output change, *first-order approximated* to $F'_K(K, L)\,\Delta K + F'_L(K, L)\,\Delta L$
- The differential: If $Q = F(K, L)$ we *define* the differential dQ of Q as $\qquad F'_K(K, L)\,dK + F'_L(K, L)\,dL.$

  (Sometimes we just write dF for dQ, identifying the "black box" with its output.)

  - The differential then obeys rules similar to the ones of derivatives: $d(Q + R) = dQ + dR$, $d(QR) = R\,dQ + Q\,dR$, and the chain rule: say, if K and L are functions of time t, then $dQ = F'_K(K, L)\,dK + F'_L(K, L)\,dL$ equals $F'_K(K, L)\,K'(t)\,dt + F'_L(K, L)\,L'(t)\,dt$.
  - The invariance property: the differential is "agnostic" as to whether a variable is free or dependent. The formula "$F'_K(K, L)\,dK + F'_L(K, L)\,dL$" remains valid if we "endogenize" K and L; just insert the new formulae for dK and dL.

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The "differential form" has a couple of advantages:

- In the formula $F'_K(K,L)\,dK + F'_L(K,L)\,dL$, you can make *simultaneous changes* in K and L.
- As the differential does not care what is "determined", then on the level curve (where $dQ = 0$) we can write the changes "without making a choice between $L = L(K)$ vs. $K = K(L)$": we have $\quad F'_K(K,L)\,dK + F'_L(K,L)\,dL = 0$.
    - This says that – up to a first-order approximation accuracy – in order to stay at the level curve, the changes in K and L must be related that way. Which we can rewrite as $F'_L(K,L)\,dL = -F'_K(K,L)\,dK$ if we want to.
    - And if we at the end of a long night of model-building find out that we want to ask the question: "if I want to reduce K by a small unit, how much must I then increase L in order to fulfil my 100 pcs order?" – then the formula $dL = -\frac{F'_K(K,L)}{F'_L(K,L)}\,dK$ remains valid unless we divide by zero.

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With more variables: $Q = F(K, L, M)$ (say): Differential now

$$F'_K(K, L, M)\, dK + F'_L(K, L, M)\, dL + F'_M(K, L, M)\, dM$$

If on a level curve AND $F'_L(K, L, M) \neq 0$, then

$$dL = \frac{-F'_K(K, L, M)}{F'_L(K, L, M)} dK + \frac{-F'_M(K, L, M)}{F'_L(K, L, M)} dM$$

We can change K and M *simultaneously* and this formula tells us $\approx$ how much L must change in order to keep constant output. If you decide to consider L as function of K and M, then the *partial* derivatives are the respective coefficients:

$$\frac{\partial L}{\partial K} = \frac{-F'_K(K, L, M)}{F'_L(K, L, M)} \qquad \frac{\partial L}{\partial M} = \frac{-F'_M(K, L, M)}{F'_L(K, L, M)}$$

(and to get the respective MRS's: switch sign.)

So: writing with differentials, you can capture both partial and simultaneous changes.

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Language/notation on this slide optional and voluntary; you can write the same content without vector notation on the exam.
(You have to know the same *content* in any case.)

Vector notation for differentials:

If $Q = F(\mathbf{x})$, then $dQ = \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\, dx_i$, which equals the dot product $\left(\frac{\partial F}{\partial x_1},\ \frac{\partial F}{\partial x_2},\ \ldots,\ \frac{\partial F}{\partial x_n}\right) \cdot \left(dx_1,\ dx_2,\ \ldots,\ dx_n\right)$ where it should really have been $\frac{\partial F}{\partial x_i}(\mathbf{x})$ (evaluation at $\mathbf{x}$) everywhere.

Typical notation: $\nabla F(\mathbf{x})$ for the *row* vector of partial first derivatives at $\mathbf{x}$, yields the matrix product form $\nabla F(\mathbf{x})\, d\mathbf{x}$.

This generalizes the univariate $F'(x)\, dx$. , and the upside-down triangle symbol saves us from using the prime symbol for neither transpose nor derivative ... the first-derivatives vector $\nabla F(\mathbf{x})$ is called the "gradient" of F at $\mathbf{x}$.

Now, by saying that the language is optional and the content is not: you are indeed expected to be able to handle the $\sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\, dx_i$ – in fact, that is the multivariate chain rule, which you should know already before Math2 – so the only optional part is, we will not by any means require you to write it as "$\nabla F(\mathbf{x})\, d\mathbf{x}$".

On to it: Given constants C and D and C^1 functions F and G. Consider the equation system

$$F(u, v, x, y, z) = C \qquad\qquad G(u, v, x, y, z) = D$$

Assume* that this equation system determines u and v as C^1 functions of (x, y, z) if part (c) below is there, there will be some "around a point where" [the equation system holds, say: where $(u, v, x, y, z) = (1, 2, 3, 4, 5)$]

Next up: a cookbook for their partial first derivatives.

In particular: for the following typical exam problem example:

(a) Differentiate the system†.

(b) Find a general expression for $\partial v/\partial y$.

(c) Approximate $v(3.1,\ 3.99,\ 5)$

* That sentence spawns some questions: Does that "assume" indeed hold true, with u and v being C^1 (and defined!) everywhere? Not necessarily, but long story short and imprecise: as long as "all our calculations make sense", the functions will be *locally* defined and C^1 and our method will give the answer. Good enough for Math2!

† "Differentiate the system" means calculate *differentials*. Norwegian has two distinct words, "deriver" for derivatives vs. "differensier" for differentials; thus, in order to convey the same information in both languages, an exam usually says e.g. "Differentiate the system (i.e., calculate differentials)" nowadays.

Cookbook for differentiating implicitly given functions, 2×2 case: $F(u, v, x, y, z) = C, \qquad G(u, v, x, y, z) = D$

Want: the partial derivatives of the implicitly given u and v. Cookbook essentially the same for any number of free variables. I chose 3 just because nothing says it must be the same number as eq's.

1. *Differentiate* the system. (Term by term or variable by variable, your choice.)

$$\begin{matrix} F'_u\,du & +F'_v\,dv & +F'_x\,dx & +F'_y\,dy & +F'_z\,dz & =0 \\ G'_u\,du & +G'_v\,dv & +G'_x\,dx & +G'_y\,dy & +G'_z\,dz & =0 \end{matrix}$$

   Everything is evaluated at (u, v, x, y, z), so these differentiated eq's could be a mess of u, du, v, dv, x, dx, y, dy, z, dz. Therefore:

2. *Identify* this as an equation system for du and dv

$$\underbrace{\begin{pmatrix} F'_u & F'_v \\ G'_u & G'_v \end{pmatrix}}_{=:\mathbf{A}} \begin{pmatrix} du \\ dv \end{pmatrix} + \underbrace{\begin{pmatrix} F'_x & F'_y & F'_z \\ G'_x & G'_y & G'_z \end{pmatrix}}_{=:\mathbf{B}} \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   where $\mathbf{A}$ and $\mathbf{B}$ depend on (u, v, x, y, z). (Nothing to "do" in this step, except catching what needs to be done – but if you do that, the rest is an algorithm.)

Matrix notation *not required* (but very useful on a slide):

3. *Solve* for du and dv: $\begin{pmatrix} du \\ dv \end{pmatrix} = -\mathbf{A}^{-1}\mathbf{B}\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$ $\quad = \mathbf{R}\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$

   and for once, this course does not ask you to check invertibility.

4. *Read off* the partial derivatives.

   Written out, we now have the form, where $(r_{ij}) = \mathbf{R} = -\mathbf{A}^{-1}\mathbf{B}$:

$$\begin{aligned} du &= r_{11}\,dx + r_{12}\,dy + r_{13}\,dz \\ dv &= r_{21}\,dx + r_{22}\,dy + r_{23}\,dz \end{aligned}$$

   and, e.g., $\frac{\partial u}{\partial z} = r_{13}$ and $\frac{\partial v}{\partial y} = r_{22}$. Note: all the r_{ij} depend on (u, v, x, y, z) (but you should *not* have any "du" left in your "dv" expression!)

Note: One equation, one determined variable redux: If we have only $F = C$ and no "v", then we get $du = -\frac{1}{F'_u}\begin{pmatrix} F'_x, & F'_y, & F'_z \end{pmatrix}\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$.
Recognize the analogues to the "$\mathbf{A}$" and "$\mathbf{B}$" matrices here!

Example problem had given a point P with coordinates $(u, v, x, y, z) = (1, 2, 3, 4, 5)$ and a question:

(c) Approximate $v(3.1,\ 3.99,\ 5)$

From $(x, y, z) = (3, 4, 5)$ to $(x + dx, y + dy, z + dz) = (3.1,\ 3.99,\ 5)$ we find $dx = 0.1$, $dy = -0.01$, $dz = 0$. We have

$$\begin{aligned} v(3.1,\ 3.99,\ 5) &= v(3, 4, 5) + \Delta v \\ &\approx 2 + dv = 2 + 0.1 \cdot r_{21}\Big|_P - 0.01 \cdot r_{22}\Big|_P \end{aligned}$$

where the $\Big|_P$ indicates that you shall *insert for the coordinates:* $(u, v, x, y, z) = (1, 2, 3, 4, 5)$.

... question: why the "2" in "$\approx 2 + dv$"?

Simple example: Apply the cookbook to calculate $\partial C/\partial G$ when (Y, C) are determined (as functions of I and G) by:

$$Y = C + I + G, \qquad C = f(Y)$$

Here f is some C^1 function with $0 < f' < 1$.

1. Differentiate: $dY = dC + dI + dG, \qquad dC = f'(Y)dY$.
2. OK, got it, we shall not solve for Y ...
   You would probably not use matrices here? Exercise: do that, just to hone your LA skills.
3. $dC = f'(Y) \cdot (dC + dI + dG)$ yields $dC = \frac{f'(Y)}{1-f'(Y)}(dI + dG)$.
   And $dY = \frac{1}{1-f'(Y)}(dI + dG)$ if you want to follow the cookbook completely, but we do not need that for $\partial C/\partial G$. Exercise: instead of inverting the coefficient matrix, or using Gaussian elimination: what could you have used from the linear algebra curriculum to get out only dC?
4. The dG coefficient in the solved-out expression for dC is $\underline{\underline{\frac{f'(Y)}{1-f'(Y)}}}$.

The "for once, this course does not ask you to check invertibility" in step 3 $\rightsquigarrow$ OK to just divide in these problems, even if I hadn't written $f' < 1$.

Example given in class: Suppose the C^1 functions $u = u(x, y, z)$ and $v = v(x, y, z)$ satisfy $u^2 + v = xy + z, \quad uv = y^2 - x^2.$

(a) Differentiate the system (i.e., calculate differentials)

(b) Find the three first-order partial derivatives of u.

(a) Calculate differentials: $2u\,du + dv = y\,dx + x\,dy + dz$ and $v\,du + u\,dv = -2x\,dx + 2y\,dy$

(b) Eliminate dv from the differentiated system, e.g. by

$$\begin{aligned} 2u\,du + dv &= y\,dx + x\,dy + dz && \mid \cdot u, \text{ then } \leftarrow + \\ v\,du + u\,dv &= -2x\,dx + 2y\,dy && -1 \end{aligned}$$

yields $(2u^2 - v)du = (uy + 2x)dx + (ux - 2y)dy + u\,dz$ and, in these particular problems you can divide w/o worrying over zeroness: $du = \underbrace{\frac{uy + 2x}{2u^2 - v}}_{=\partial u/\partial x}\,dx + \underbrace{\frac{ux - 2y}{2u^2 - v}}_{\partial u/\partial y}\,dy + \underbrace{\frac{u}{2u^2 - v}}_{\partial u/\partial z}\,dz$

more on the same example, and notes:

So we can just read off the derivatives as indicated:

$$\frac{\partial u}{\partial x} = \underline{\underline{\frac{uy + 2x}{2u^2 - v}}}, \qquad \frac{\partial u}{\partial y} = \underline{\underline{\frac{ux - 2y}{2u^2 - v}}}, \qquad \frac{\partial u}{\partial z} = \underline{\underline{\frac{u}{2u^2 - v}}}$$

- Though the cookbook would want you to solve for du and dv, the question only asks for the first-order partial derivatives of u, and so dv is not needed. (Just make sure you have eliminated it!)
- I asked a Q: if z were not a variable, but a constant: would that affect u'_x?
  A: No; a partial change in x is as if the other free variables – in this case y and z – were treated as constants.
  Note: This has nothing to do with z not appearing in the expressions!

Then: *When and how can we speed up?* (Reading the problem helps!)
(And in time squeeze: get method right! Maybe not give *highest* priority to debugging expressions like slide 17...?)

- (Already done more than once:) If "part (b)" only asks for partial derivatives *of* one variable (say, u), then solving for du gives you what you need. You can use Cramér if you like.
- If "part (b)" only asks for partial derivatives *with respect to* one free variable (say, x), then put $dy = dz = 0$
(the other *free* var's only! Do not delete the diff. of the *dependent* dv!)
  - Leave the answer to part (a) as-is, this is only for (b). In the example, when starting at (b) you can simplify to
  $2u\,du + dv = y\,dx \quad | \quad \cdot u$, then add ($+$) to the equation below, which is multiplied by -1
  $v\,du + u\,dv = -2x\,dx$
  and so $(2u^2 - v)du = (uy + 2x)dx$ and $\frac{\partial u}{\partial x} = \frac{uy+2x}{2u^2-v}$ as we saw; then $\frac{\partial v}{\partial x} = y - 2u\frac{\partial u}{\partial x} = y - 2u\frac{uy+2x}{2u^2-v} = -\frac{yv+4xu}{2u^2-v}$
- If it asks for the derivatives *merely at a point,* then insert for point coord's at the beginning of "part (b)". But beware ...

cont'd next slide

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- cont's: derivatives merely at point. Say, the example gives the point where $(u, v, x, y, z) = (1, 5, 2, 3, 0)$ and the question: "(b) Find $\frac{\partial u}{\partial x}(2, 3, 0)$ and $\frac{\partial v}{\partial x}(2, 3, 0)$."
  - Again, the answer to part (a) should be left as-is. and again, only derivatives wrt. x are asked, so put $dy = dz = 0$. Furthermore, you can now insert point coordinates and work with the system $2\,du + dv = 3\,dx$ and $5\,du + dv = -4\,dx$. Subtracting, we have $-3\,du = 7\,dx$ and $\frac{\partial u}{\partial x}(2, 3, 0) = -7/3$; then $\frac{\partial v}{\partial x}(2, 3, 0) = 3 - 2\frac{\partial u}{\partial x}(2, 3, 0) = 3 + 14/3 = 23/3$.
- *Beware:* problem might say "around the point where $(u, v, x, y, z) = (1, 5, 2, 3, 0)$" and still ask for a general expression for $\frac{\partial u}{\partial y}$. Then the answer has "letters", ("$\frac{ux-2y}{2u^2-v}$")
  - Why give coordinates then? Consider one eq., $x^2 + y^2 = 2$. That does not define $y = y(x)$ (the circle is not a function graph). But "around the point where $(x, y) = (1, 1)$", it *does* define a function graph: the upper half-circle.

New example for Wednesday: The equation system

$$se^{y-x} + \ln(2t+y) + x = 3, \qquad ye^{-x} + stxy + t^2 = e^{-1}$$

defines continuously differentiable functions $x = x(s,t)$ and $y = y(s,t)$ around the point where $(s,t,x,y) = (2,0,1,1)$. (You shall not show this.)

(a) Differentiate the system (i.e., calculate differentials).

(b) Find a general expression for $\frac{\partial x}{\partial s}$.
Alternative question (b'): Calculate $\frac{\partial x}{\partial s}(2,0)$.

(a): $e^{y-x}\,ds + se^{y-x}(dy - dx) + \frac{1}{2t+y}(2\,dt + dy) + dx = 0$ and
$e^{-x}\,dy - ye^{-x}\,dx + txy\,ds + sxy\,dt + sty\,dx + stx\,dy + 2t\,dt = 0$.

(b): Only asked for $\partial x/\partial s$, so put $dt = 0$. Collect/reorder terms:
$(1 - se^{y-x})\,dx + (se^{y-x} + \frac{1}{2t+y})\,dy = -e^{y-x}\,ds$ and
$(st - e^{-x})y\,dx + (stx + e^{-x})\,dy = -txy\,ds$. Now eliminate dy – or use Cramér for dx:

new example cont'd: Cramér for dx on the differentiated system

$$\begin{array}{lllll} (1-se^{y-x})\,dx & + & (se^{y-x}+\frac{1}{2t+y})\,dy & = & -e^{y-x}\,ds \\ (st-e^{-x})y\,dx & + & (stx+e^{-x})\,dy & = & -txy\,ds \end{array} \qquad \text{(D)}$$

yields $dx = \dfrac{\begin{vmatrix} -e^{y-x}\,ds & se^{y-x}+\frac{1}{2t+y} \\ -txy\,ds & stx+e^{-x} \end{vmatrix}}{\begin{vmatrix} 1-se^{y-x} & se^{y-x}+\frac{1}{2t+y} \\ (st-e^{-x})y & stx+e^{-x} \end{vmatrix}} = \dfrac{\partial x}{\partial s}\,ds$ where

$$\frac{\partial x}{\partial s} = \underline{\underline{\frac{-(stx+e^{-x})e^{y-x}+txy\cdot\left(se^{y-x}+\frac{1}{2t+y}\right)}{(1-se^{y-x})(stx+e^{-x})-(se^{y-x}+\frac{1}{2t+y})\cdot(st-e^{-x})y}}}$$

(b'): If only asked for $\frac{\partial x}{\partial s}(2,0)$: Insert $(s,t,x,y)=(2,0,1,1)$ into (D), which then simplifies to $(1-2)dx+(2+1)dy=-ds$ and $-e^{-1}dx+e^{-1}dy=0$. From the latter, $dy=dx$ and so the former says $2dx=-ds$. Thus, $\frac{\partial x}{\partial s}(2,0)=\underline{\underline{-1/2}}$.

Finally, case $n > 2$ covered:

- Setup: n equations[‡] $f_1(\mathbf{u}, \mathbf{x}) = C_1, \ldots, f_n(\mathbf{u}, \mathbf{x}) = C_n$ determining $\mathbf{u} \in \mathbb{R}^n$ as n functions $u_1(\mathbf{x}), \ldots, u_n(\mathbf{x})$.
- Here, $\mathbf{x}$ are m variables, m could be any natural number.
- Cookbook: Differentiating this system gives the form

$$\mathbf{A}\,d\mathbf{u} + \mathbf{B}\,d\mathbf{x} = \mathbf{0} \quad \text{so that} \quad d\mathbf{u} = -\mathbf{A}^{-1}\mathbf{B}\,d\mathbf{x}$$

  where $\mathbf{A} = (a_{ij})_{i,j}$, $a_{ij} = \frac{\partial f_i}{\partial u_j}(\mathbf{u}, \mathbf{x})$ and $\mathbf{B} = \left(\frac{\partial f_i}{\partial x_j}(\mathbf{u}, \mathbf{x})\right)_{i,j}$.

  (Just a bigger linear equation system. n equations, n unknowns $du_1, \ldots, du_n$.)
- Partial derivatives: $\partial u_i / \partial x_j =$ element (i, j) of $\mathbf{R} := -\mathbf{A}^{-1}\mathbf{B}$.
- (Matrix notation: You might encounter in the literature – though certainly not on a Math2 exam – formulae like $\frac{\partial \mathbf{u}}{\partial \mathbf{x}} = -\left(\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right)^{-1} \frac{\partial \mathbf{f}}{\partial \mathbf{x}}$. Or with arrows: $\frac{\partial \vec{u}}{\partial \vec{x}} = -\left(\frac{\partial \vec{f}}{\partial \vec{u}}\right)^{-1} \frac{\partial \vec{f}}{\partial \vec{x}}$.)

  (And if you see the phrase "Jacobian" matrix, it is a matrix of *first-order* derivatives. Not the Hessian.)

[‡] Last-minute change from "F" to "f" due to the last bullet item: $\mathbf{f}$ is a column vector (not a matrix) of functions.