Elasticities as differentials; the elasticity of substitution

Elasticities as differentials.

Derivative: the instantaneous rate of change in units per units.
Elasticity: the instantaneous rate of change in "percent per percent". That is, if you like: *on logarithmic scale.*

The elasticity $\mathrm{El}_x f(x)$ equals $\frac{x}{f(x)} f'(x)$, but it also equals $\frac{d \ln f(x)}{d \ln x}$ as long as f and x both > 0. More generally: $\frac{d \ln |f(x)|}{d \ln |x|}$ if $x \neq 0$. (Verify: calculate the differentials! $d \ln |f| = \frac{1}{f}\, df$, and $d \ln |x| = \frac{1}{x}\, dx$. Divide: $\frac{x}{f} \cdot \frac{df}{dx}$ *et voilà.*) ... does "dividing differentials into derivatives" look like a cheat? Kind of, so just take our word that it works. Someone has done the job and proven it.

Application (that is: just motivation, not curriculum): A regression $y = \alpha + \sum_i \beta_i x_i + \epsilon$, estimates how much y will change when each x_i changes. $\rightsquigarrow$ derivatives.
OTOH, logging everything $\rightsquigarrow$ elasticities.
To see that: put $\upsilon = \ln y$, $\xi_i = \ln x_i$. The regression $\upsilon = \alpha + \sum_i \beta_i \xi_i + \epsilon$ estimates how much $\ln y$ will change when $\ln x_i$ changes; that is, $\approx$ "how many percent y changes when x_i changes with a percent".

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The d ln formulation gives most elasticity rules easily:

- $\mathrm{El}_x(Ax^r) = \frac{d\ \ln A + r\ d\ln x}{d\ln x} = r$
- $\mathrm{El}_x(e^{bx}) = \frac{d(bx)}{d\ln x} = \frac{b\ dx}{x^{-1}\ dx} = bx$. (But $\mathrm{El}(x^x) \neq x!$)
- If $h(x) = f(x)g(x)$, then $\mathrm{El}_x h = \frac{d\ \ln f + d\ln g}{d\ln x} = \mathrm{El}_x f + \mathrm{El}_x g$.
- If $h(x) = g(f(x))$, the chain rule works like for derivatives:
  $\mathrm{El}_x h = \frac{d\ \ln g}{d\ \ln f} \cdot \frac{d\ln f}{d\ln x} = \mathrm{El}_y g\big|_{y=f(x)} \mathrm{El}_x f$.
  - On notation: e.g. "$\mathrm{El}_x g(f(x))$" ... what does that mean?
    Compare derivatives: $g'(f(x))$ vs. $\frac{d}{dx} g(f(x))$. Two distinct symbols d and ∂ allows formulae like $\frac{d}{dK} F(K, L(K)) = \frac{\partial F}{\partial K} + \frac{\partial F}{\partial L}\frac{dL}{dK}$.
    But, does $\mathrm{El}_K F(K, L(K))$ mean "partial" or "total" elasticity?
    (*Typically:* partial wrt. 1st variable, but ... economists sometimes think otherwise)
- $\mathrm{El}_x(f+g) = \frac{d\ \ln(f+g)}{d\ln x} = \frac{df + dg}{f+g} \cdot \frac{1}{d\ \ln x} = \frac{f \cdot (df\ /\ f) + g \cdot (dg\ /\ g)}{(f+g) \cdot d\ \ln x}$
  $= \frac{f \cdot \mathrm{El}_x f + g \cdot \mathrm{El}_x g}{f+g}$ (value-weighted avg. of the elasticities.)

Ex.: $\frac{d\ln(f(x)^{g(x)})}{d\ln x} = \frac{d[g\ln f]}{d\ln x} = g\frac{d\ln f}{d\ln x} + \frac{dg}{d\ln x}\ln f = g\Big[\mathrm{El}_x f + \ln f \cdot \mathrm{El}_x g\Big]$

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Back to $\frac{\partial L}{-\partial K} = \frac{\partial F}{\partial K}\Big/\frac{\partial F}{\partial L}$ on the level curve $F(K, L) = C$.

- Ask instead: "if I want to reduce K by a *percentage*, how many *percent* will I have to increase L in order to keep F constant?"
- A: Relative increase in in L divided by relative reduction in K: $\frac{\partial L}{L}\Big/\left(-\frac{\partial K}{K}\right) = K\frac{\partial F}{\partial K}\Big/\left(L\frac{\partial F}{\partial L}\right) = \frac{K}{F}\cdot\frac{\partial F}{\partial K}\Big/\left(\frac{L}{F}\cdot\frac{\partial F}{\partial L}\right) = \mathrm{El}_K F/\mathrm{El}_L F$
- Ex.: Cobb–Douglas $K^a L^b$, yields a/b.
  - Note for later: The factor use *ratio* L/K increases by $(1 + a/b)$ (unit: percent per percent reduction in K)

That was the elasticity of one function given by one equation.
What about of functions given by equation *systems?*
From, e.g., $\frac{\partial v}{\partial y}$, get $\mathrm{El}_y v = \frac{y}{v}\cdot\frac{\partial v}{\partial y}$. Or phrase the differentiated system as $\mathbf{S}\begin{pmatrix} du\,/\,u \\ dv\,/\,v \end{pmatrix} = -\mathbf{Q}\begin{pmatrix} dx\,/\,x \\ dy\,/\,y \\ dz\,/\,z \end{pmatrix}$ and solve for e.g. $\dfrac{dv}{v}$.

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The elasticity of substitution: $F(K, L)$ defined for $K > 0$, $L > 0$, fixed level curve $F(K, L) = C$. (Level curve assumed strictly decreasing. Not unreasonable for isoquants ... but no Leontief ....)

- Composition of factors: A point on the level curve $\longleftrightarrow$ a ratio L/K.
- Substitution: a percentwise change in L/K (cf. previous slide).
- Q: if we move along the level curve so much that the MRS changes by one percent; how much (in percent) does the ratio L/K change? (Who would come up with that question? Blame economics! A one percent change in price (ratio) ...)
- This is the *elasticity of substitution* σ_{LK} between K and L:

$$\sigma_{LK} = \frac{d \ln(L/K)}{d \ln[\text{MRS}]} = \frac{d \ln(L/K)}{d \ln(F'_K/F'_L)}$$

  - Formula in the book (under harder problems) and last slide.
  - Examples next slide: will manipulate differentials.
- Note: $\sigma_{KL} = \sigma_{LK}$. (Why? ... thus language "between ... and")

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The elasticity of substitution. *Example:* Cobb–Douglas.
When $F'_K = aF/K$ and $F'_L = bF/L$, the ratio $\frac{d\ \ln(L/K)}{d\ \ln(F'_K/F'_L)}$ is 1.

(Why? $d\ \ln(aL/bK) = d\,[\ln(a/b) + \ln(L/K)] = d\ \ln(L/K)$ since the differential of a constant is zero.)

So – with adaptation at MRS = price ratio and still assuming output fixed at C – a one percent change in price ratio $w/p \rightsquigarrow$ a one percent change in factor use ratio L/K.)

Example: The CES functions $F(K, L) = A(\alpha K^{-Q} + \beta L^{-Q})^{-m/Q}$.
Need partial derivatives. The differential of F is $dF =$

$$\frac{-m}{Q} \cdot \frac{F}{\alpha K^{-Q} + \beta L^{-Q}}\, d[\alpha K^{-Q} + \beta L^{-Q}] = \frac{mF}{\alpha K^{-Q} + \beta L^{-Q}} \cdot \left[\frac{\alpha\, dK}{K^{Q+1}} + \frac{\beta\, dL}{L^{Q+1}}\right]$$

so $F'_K/F'_L = \frac{\alpha}{\beta} \cdot (L/K)^{Q+1}$. Apply log and differential:
$d\ \ln(\frac{\alpha}{\beta} \cdot (L/K)^{Q+1}) = 0 + (Q + 1)\, d\ln(L/K)$.

Elasticity of substitution: $\frac{d\ln(L/K)}{(Q+1)\, d\ln(L/K)} = \frac{1}{Q+1}$.

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Exercise: Let $g' > 0$ everywhere. Show that σ_{LK} is the same for $g(F(K,L))$ on the level curve $g(C)$ as for F on level curve C.

Limits exercise: Show that $\lim_{Q\to 0}$ CES is a Cobb–Douglas.

Linear algebra exercise: As mentioned, there is a formula in the book (and you are free to use that!); σ_{KL} happens to equal $\frac{F'_K F'_L \cdot (KF'_K + LF'_L)}{KL \cdot B}$, where $B = -\left[(F'_L)^2 F''_{KK} - 2F'_K F'_L F''_{KL} + (F'_K)^2 F''_{LL}\right]$

Find a t such that this B equals the determinant $\begin{vmatrix} t & F'_K & F'_L \\ F'_K & F''_{KK} & F''_{KL} \\ F'_L & F''_{KL} & F''_{LL} \end{vmatrix}$
(all primes denote derivatives! No transposes here.)
Post-lecture update: Calculate, see that $t = 0$ fits.

This – inserted $t = 0$ – is called the *bordered Hessian determinant* of F.
Note that the lower–right 2×2 block is the Hessian (i.e., has the second-order derivatives), and then there is a border of first-order derivatives and that top–left element.

Fact (4200/Math3): A C^2 function of two variables is *quasiconcave* iff its bordered Hessian determinant is *nonnegative everywhere.*

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Finally, that "expression for" re-clarified: [disclaimer: a commitment for *2018*!]

- If you are asked to *solve* an equation system, we want the truth, the whole truth and nothing but the truth. (Likewise for, e.g., "find the points which satisfy the Kuhn–Tucker conditions".)
- If asked: "Find an expression for the inverse of $\mathbf{A}_t$", then using the formula to get, say, $\frac{3}{t(1-t)}\begin{pmatrix} 2 & -3 & 4t \\ t & 9 & -3 \\ 0 & 0 & 2 \end{pmatrix}$, suffices.
  - It is correct whenever the inverse exists – with or without any "as long as $t \notin \{0, 1\}$" (which won't hurt ...)
- Sometimes we allow for slightly less. Motivation: MRS for $F(K, L) = (KL)^a$, is $(L/K)^{1-a}$. That is also the MRS for $G(K, L) = 1 + (F(K, L) - 1)^3$. Why? Formula: $\frac{3(F-1)^2}{3(F-1)^2} \cdot (L/K)^{1-a}$. *Should "expression for" expect you to treat case* $F = 1$ *separately?*
  - For derivatives of implicitly given functions: no. Get an expression. (Zero denominator? Do not bother to check.)
  - Elasticity of substitution: same. Just get an expression.
- Of course, you could be asked explicitly to compute a limit!