Suggested Solutions: ECON3120/4120 Mathematics 2, Delayed exam Autumn 2025

January 2026

Problem 1 Constrained Optimization (a)

$$\ln(x-y) - e^{xyz} - \lambda_1(x+y+z-6) - \lambda_2(2x-y-b)$$

(b)

$$\frac{1}{x-y} - yze^{xyz} - \lambda_1 - 2\lambda_2 = 0 \tag{1}$$

$$-\frac{1}{x-y} - zxe^{xyz} - \lambda_1 + \lambda_2 = 0 \tag{2}$$

$$-yxe^{xyz} - \lambda_1 = 0 \tag{3}$$

$$x+y+z = 6 \tag{4}$$

$$2x-y = b \tag{5}$$

(c) It is possible to prove that the unique Lagrange-point solves the maximization problem.

Using the claim in the hint that $F(x,y,z)$ inherit the properties of $f(x,y) = \ln(x-y)$ the function $F(x,y,z)$ would be concave, while the constraints are both linear (hence concave and convex). (The function $\tilde{F}(x,y,z) = \ln(x-y) - e^{x+y+z}$ is indeed concave, but in the final exercise, F is not.) For the concavity of $f(x,y)$ see below.

This hint/assumption was confusing as $F(x,y,z)$ is in fact not concave and more a advanced argument is required to correctly prove that the unique Lagrange-point actually solves the problem.

As for the properties for $f(x,y)$:

$$f'_x(x,y) = \frac{1}{x-y}$$

$$f'_y(x,y) = -\frac{1}{x-y}$$

$$f''_{xx}(x,y) = -\frac{1}{(x-y)^2} < 0$$

$$f''_{yy}(x,y) = -\frac{1}{(x-y)^2} < 0$$

$$f''_{xy}(x,y) = \frac{1}{(x-y)^2}$$

Then:

$$\left(-\frac{1}{(x-y)^2}\right)\left(-\frac{1}{(x-y)^2}\right) - \left[\frac{1}{(x-y)^2}\right]^2 = 0$$

Thus f is concave for all feasible (x, y). The problem stated that we then could assume that also $j(x, y, z)$ is concave. Sum of concave functions is concave, so F would be concave.

Using the hint this way, gives of course full score. Due to the confusion, wrong answers on 1c) and 1d) got very low weight while an answer using the hint as above would give full weight.

(d) By the envelope theorem:

$$L^*(x^*, y^*, z^*) = \ln(x^* - y^*) - e^{x^* y^* z^*} - \lambda_1(x^* + y^* + z^* - 6) - \lambda_2(2x^* - y^* - b)$$

Thus:

$$\frac{\partial L^*}{\partial b} = \lambda_2$$

Problem 2 Integration

(a) We are given the integral:

$$I(x) = \int_x^{x^3} x^2 e^t \, dt$$

Since the integration is with respect to t, the term x^2 is treated as a constant and can be moved outside the integral:

$$I(x) = x^2 \int_x^{x^3} e^t \, dt$$

The antiderivative of e^t is simply e^t. We apply the Fundamental Theorem of Calculus:

$$I(x) = x^2 \left[e^t\right]_x^{x^3}$$

Substituting the upper and lower limits:

$$I(x) = x^2 \left(e^{x^3} - e^x\right)$$

Shown.

(b) We must verify that $I(x)$ satisfies:

$$I'(x) - \left(\frac{2}{x} + 3x^2\right) I(x) = (3x^4 - x^2)e^x$$

First, we compute the derivative $I'(x)$ using the product rule on $I(x) = x^2 e^{x^3} - x^2 e^x$:

$$\begin{aligned} I'(x) &= \frac{d}{dx}(x^2 e^{x^3}) - \frac{d}{dx}(x^2 e^x) \\ &= \left(2xe^{x^3} + x^2 e^{x^3} \cdot 3x^2\right) - \left(2xe^x + x^2 e^x\right) \\ &= 2xe^{x^3} + 3x^4 e^{x^3} - 2xe^x - x^2 e^x \end{aligned}$$

Next, we expand the term $\left(\frac{2}{x} + 3x^2\right) I(x)$:

$$\begin{aligned} \left(\frac{2}{x} + 3x^2\right)(x^2 e^{x^3} - x^2 e^x) &= \left(\frac{2}{x} + 3x^2\right) x^2 (e^{x^3} - e^x) \\ &= (2x + 3x^4)(e^{x^3} - e^x) \\ &= 2xe^{x^3} - 2xe^x + 3x^4 e^{x^3} - 3x^4 e^x \end{aligned}$$

Now, we perform the subtraction for the Left Hand Side (LHS) of the differential equation:

$$\begin{aligned} \text{LHS} &= I'(x) - \left(\frac{2}{x} + 3x^2\right) I(x) \\ &= \left(\mathbf{2xe^{x^3}} + \mathbf{3x^4 e^{x^3}} - \mathbf{2xe^x} - x^2 e^x\right) \\ &\quad - \left(\mathbf{2xe^{x^3}} - \mathbf{2xe^x} + \mathbf{3x^4 e^{x^3}} - 3x^4 e^x\right) \end{aligned}$$

Canceling the matching terms (bolded):

$$\text{LHS} = -x^2 e^x - (-3x^4 e^x) = 3x^4 e^x - x^2 e^x = (3x^4 - x^2)e^x$$

This matches the Right Hand Side. **Shown.**

(c) We are asked to solve the linear differential equation:

$$I'(x) - \left(\frac{2}{x} + 3x^2\right) I(x) = (3x^4 - x^2)e^x \tag{6}$$

The general solution to a linear differential equation is the sum of the particular solution and the solution to the homogeneous equation:

$$I(x) = I_p(x) + F(x)$$

From the result verified in **Part (a)** and **Part (b)**, we know that a particular solution to this equation is:

$$I_p(x) = x^2(e^{x^3} - e^x)$$

The corresponding homogeneous equation is:

$$F'(x) - \left(\frac{2}{x} + 3x^2\right) F(x) = 0$$

This is a homogeneous linear differential equation of the form $F'(x) + P(x)F(x) = 0$, where:

$$P(x) = -\left(\frac{2}{x} + 3x^2\right)$$

We use the general formula for the solution of a homogeneous linear equation:

$$F(x) = Ce^{-\int P(x)\,dx}$$

Substituting $P(x)$:

$$-\int P(x)\,dx = -\int \left[-\left(\frac{2}{x} + 3x^2\right)\right] dx = \int \left(\frac{2}{x} + 3x^2\right) dx$$

Evaluating the integral:

$$\int \left(\frac{2}{x} + 3x^2\right) dx = 2\ln(x) + x^3$$

(Note: We absorb the integration constant into C). Now, substitute back into the formula for $F(x)$:

$$F(x) = Ce^{2\ln(x)+x^3}$$

Using laws of exponents ($e^{A+B} = e^A e^B$ and $e^{2\ln x} = e^{\ln x^2} = x^2$):

$$F(x) = C \cdot x^2 \cdot e^{x^3}$$

The general solution is:

$$I(x) = I_p(x) + F(x)$$

$$I(x) = \left[x^2(e^{x^3} - e^x)\right] + Cx^2e^{x^3}$$

We use the condition $I(1) = 0$ to find C:

$$0 = \left[1^2(e^{1^3} - e^1)\right] + C(1)^2e^{1^3}$$

$$0 = (e - e) + Ce$$

$$0 = Ce \implies C = 0$$

Thus, the final solution is:

$$I(x) = x^2(e^{x^3} - e^x)$$

Problem 3 Linear Algebra

Problem Statement

Consider the matrices:

$$A_a = \begin{pmatrix} 1 & 0 & 3 \\ 0 & a & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

where a is an arbitrary real constant.

Solution

Determine whether the matrix products A_aC, $b'A_a$, and Cb are well-defined. If so, compute them.

1. Product A_aC:

The matrix A_a is size 3×3 and C is size 3×2. Since the number of columns in A_a (3) equals the number of rows in C (3), the product is **well-defined**.

$$A_aC = \begin{pmatrix} 1 & 0 & 3 \\ 0 & a & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1(1)+0(0)+3(0) & 1(0)+0(1)+3(0) \\ 0(1)+a(0)+1(0) & 0(0)+a(1)+1(0) \\ 2(1)+0(0)+1(0) & 2(0)+0(1)+1(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & a \\ 2 & 0 \end{pmatrix}$$

2. Product $b'A_a$:

Assuming b' denotes the transpose b^T, b' is size 1×3. A_a is size 3×3. Since the inner dimensions match ($3 = 3$), the product is **well-defined**.

$$b' = \begin{pmatrix} 4 & 2 & 3 \end{pmatrix}$$

$$b'A_a = \begin{pmatrix} 4 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 3 \\ 0 & a & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

Calculating the entries:

$$\begin{aligned} &\text{Entry } 1 = 4(1) + 2(0) + 3(2) = 4 + 6 = 10 \\ &\text{Entry } 2 = 4(0) + 2(a) + 3(0) = 2a \\ &\text{Entry } 3 = 4(3) + 2(1) + 3(1) = 12 + 2 + 3 = 17 \end{aligned}$$

$$b'A_a = \begin{pmatrix} 10 & 2a & 17 \end{pmatrix}$$

3. Product Cb:

C is size 3×2 and b is size 3×1. The number of columns in C (2) does not equal the number of rows in b (3). Therefore, the product is **undefined**.

Compute the determinants of A_a and A_a^2.

To find $\det(A_a)$, we perform cofactor expansion along the second column (since it has two zeros):

$$\det(A_a) = \begin{vmatrix} 1 & 0 & 3 \\ 0 & a & 1 \\ 2 & 0 & 1 \end{vmatrix} = a \cdot (-1)^{2+2} \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = a(1(1) - 3(2)) = a(1 - 6) = -5a.$$

Using the property $\det(M^2) = (\det(M))^2$:

$$\det(A_a^2) = (\det(A_a))^2 = (-5a)^2 = 25a^2.$$

For what value(s) of a does A_a not have an inverse?

A matrix is non-invertible (singular) if and only if its determinant is zero.

$$\det(A_a) = 0 \implies -5a = 0 \implies a = 0.$$

Thus, A_a does not have an inverse when $a = 0$.

For values of a where A_a does not have an inverse, does the equation system $A_a x = b$ have no solution or many solutions? Explain.

We set $a = 0$. The augmented matrix $[A_0|b]$ is:

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 2 & 0 & 1 & 3 \end{pmatrix}$$

We perform Gaussian elimination. Subtract 2 times Row 1 from Row 3 ($R_3 \to R_3 - 2R_1$):

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -5 & -5 \end{pmatrix}$$

Looking at the second row, we have the equation $0x + 0y + 1z = 2$, which implies $z = 2$. Looking at the third row, we have $0x + 0y - 5z = -5$, which implies $z = 1$.

Since z cannot be both 2 and 1 simultaneously ($2 \neq 1$), we have a contradiction. Therefore, the system has **no solution**.

For $a = 1$, solve the equation system $A_a x = b$ using Gaussian elimination.

Set $a = 1$. The augmented matrix is:

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 2 & 0 & 1 & 3 \end{pmatrix}$$

Step 1: Eliminate the 2 in position (3,1) by performing $R_3 \to R_3 - 2R_1$:

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -5 & -5 \end{pmatrix}$$

Step 2: Scale R_3 by $-\frac{1}{5}$ to get a pivot of 1 ($R_3 \to -\frac{1}{5}R_3$):

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

This is now in row echelon form. We perform back-substitution:

1. From R_3: $z = 1$. 2. From R_2: $y + z = 2 \implies y + 1 = 2 \implies y = 1$. 3. From R_1: $x + 3z = 4 \implies x + 3(1) = 4 \implies x = 1$.

The solution is:

$$x = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Problem 4 Differential and Approximation

(a) Differentiating the system $F = 0$ and $G = 0$:

$$dF = F'_s ds + F'_t dt + F'_x dx + F'_y dy = 0$$
$$dG = G'_s ds + G'_t dt + G'_x dx + G'_y dy = 0$$

Partial derivatives evaluated at $(s, t, x, y) = (2, 1, 1, 1)$:

$$F'_s = x^2/s = 1^2/2 = \mathbf{1/2}$$
$$F'_t = y^3 = 1^3 = \mathbf{1}$$
$$F'_x = 2x\ln(s) = 2(1)\ln(2) = \mathbf{2}\ln(\mathbf{2})$$
$$F'_y = 3ty^2 = 3(1)(1)^2 = \mathbf{3}$$
$$G'_s = t^2 y = 1^2(1) = \mathbf{1}$$
$$G'_t = 2tsy = 2(1)(2)(1) = \mathbf{4}$$
$$G'_x = 3x^2 = 3(1)^2 = \mathbf{3}$$
$$G'_y = t^2 s = 1^2(2) = \mathbf{2}$$

Substituting:

$$\frac{1}{2}ds + 1dt + 2\ln(2)dx + 3dy = 0$$
$$1ds + 4dt + 3dx + 2dy = 0$$

(b) To find $\frac{\partial x}{\partial s}$ and $\frac{\partial y}{\partial s}$, we set $dt = 0$. Rearranging the system for dx and dy:

$$2\ln(2)dx + 3dy = -\frac{1}{2}ds$$
$$3dx + 2dy = -1ds$$

Dividing by ds:

$$\begin{pmatrix} 2\ln(2) & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \end{pmatrix} = \begin{pmatrix} -1/2 \\ -1 \end{pmatrix}$$

Determinant of the coefficient matrix A is $|A| = 4\ln(2) - 9$. By Cramer's Rule:

$$\frac{\partial x}{\partial s} = \frac{1}{|A|}\begin{vmatrix} -1/2 & 3 \\ -1 & 2 \end{vmatrix} = \frac{1}{4\ln(2) - 9}(-1 + 3) = \frac{\mathbf{2}}{\mathbf{4}\ln(\mathbf{2}) - \mathbf{9}}$$

Problem 5 Dynamic Programming (Hotelling's Rule)

The problem is $\max_{q_t} \sum_{t=0}^{T-1} \beta^t u(q_t)$, where $u(q_t) = aq_t - \frac{b}{2}q_t^2$.

(a) The Bellman equation for the value function $V_t(S_t)$ is:

$$V_t(S_t) = \max_{0 \leq q_t \leq S_t} \{u(q_t) + \beta V_{t+1}(S_{t+1})\}$$

The terminal condition is $V_T(S) = 0$.

(b) Assuming an interior solution ($0 < q_t < S_t$), the First-Order Condition (FOC) is found by differentiating the expression inside the max operator with respect to the choice variable q_t:

$$\frac{\partial}{\partial q_t} \left(u(q_t) + \beta V_{t+1}(S_t - q_t)\right) = 0$$

$$u'(q_t) = \beta V'_{t+1}(S_{t+1})$$

(c) For the terminal problem $V_T(S) = 0$ so the Bellman equation is

$$V_{T-1}(S_{T-1}) = \max_{0 \leq q_{T-1} \leq S_{T-1}} \{u(q_{T-1}) + \beta V_T(S_T)\} \qquad = \max_{0 \leq q_{T-1} \leq S_{T-1}} \{u(q_{T-1})\}$$

And since $u(q_{T-1})$ is increasing the optimum is $q_{T-1} = S_{T-1}$