University of Oslo / Department of Economics

ECON3120/4120 Mathematics 2: on the autumn 2019 postponed exam 2020-01-16

- *Standard disclaimer:* A note like this is not suited as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
  - The document reflects what was expected in that particular semester, and which may not be applicable to future semesters. In particular, what tests one is required to perform before answering «no conclusion» may not apply for later.
- Weighting: At the discretion of the committee (and in case of appeals: the new grading committee). The committee might want to consider the next two bullet items.

  The problem set was written with the intention that a uniform weighting over letter-enumerated items should be a *feasible* choice, and this has been communicated.
- *Special considerations for 2019: new exam format.* See the guidelines for the ordinary exam.
- *Addendum after grading:* Graded much alike pre-2019 conversion default, to the extent those thresholds had any neighbouring score (only 8 papers in total).

Problems (restated as given) and solutions and annotations (boxed) follow:

Problem 1

Take for granted that this system determines $u = u(s)$ and $v = v(s)$:

$$v \cdot e^{3u} + u^2 + s = 0$$
$$3v \cdot e^{3u} + 2u = C \qquad (C \text{ a constant})$$

(a) Differentiate the system; i.e., calculate (total) differentials.

(b) *Use the differentiated system* to find an expression for $u'(s)$. (You shall use the differentiated system – there is *no score* for eliminating v from the *original* system.)

Problem 1 solved:

(a) Differentiating:

$$\big(3ve^{3u} + 2u\big)\, du + e^{3u}\, dv + ds = 0$$
$$\big(9ve^{3u} + 2\big) du + 3e^{3u}\, dv = 0$$

(b) To eliminate dv, scale the first equation by 3 and subtract: $0 = 3(3ve^{3u} + 2u)\, du + 3ds - (9ve^{3u} + 2)du$ which equals $(6u - 2)du + 3ds$. Therefore, $u'(s) = \dfrac{3}{2 - 6u}$

Problem 2 Let $\mathbf{A} = \begin{pmatrix} t & 3 & 1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ t & 0 & 0 & t \end{pmatrix}$, $\mathbf{M} = \begin{pmatrix} t^2+10 & \beta & 4 & t^2 \\ t-2 & 3 & 0 & t \\ 4 & 0 & 2 & 0 \\ t^2 & t & 0 & \gamma \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} t \\ 0 \\ 1 \\ t \end{pmatrix}$

(a)
- Calculate $3\mathbf{A} - \mathbf{A}'$ (The prime symbol denotes transpose.)
- Not all of $\mathbf{Ab}$, $\mathbf{bM}$, $\mathbf{b}'\mathbf{b}$, $\mathbf{bb}'$ are well-defined. Find one which is *not.*

(b) Fact: $\mathbf{AA}' = \mathbf{M}$ for some β and γ (which might depend on t). Find β and γ. (You are not asked to check the other fourteen elements.)

(c) Calculate the determinant of $\mathbf{A}$ and of $\frac{1}{2}\mathbf{AA}'$. (They depend on t.)

(d) Decide for what values of t the equation system $\mathbf{Ax} = \mathbf{b}$ (where $\mathbf{x}$ is the vector of unknowns) has no solution, one solution or more than one solution.

Problem 2 solved:

(a)
- $$3\mathbf{A} - \mathbf{A}' = \begin{pmatrix} 3t & 9 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 0 & 3 & 3 & 0 \\ 3t & 0 & 0 & 3t \end{pmatrix} - \begin{pmatrix} t & 1 & 0 & t \\ 3 & -1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & t \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2t & 8 & 3 & -t \\ 0 & -2 & 2 & 0 \\ -1 & 2 & 2 & 0 \\ 3t & 0 & 0 & 2t \end{pmatrix}}}$$
- $\mathbf{bM}$ would be 4×1 by 4×4 and is not well-defined. (The others are.)

(b) $\beta = (t,\ 3,\ 1,\ 0) \cdot (1,\ -1,\ 1,\ 0) = t - 3 + 1 = \underline{\underline{t-2}}$.
$\gamma = (t,\ 0, 0, t) \cdot (t,\ 0, 0, t) = \underline{\underline{2t^2}}$.

(c) Cofactor expanding $|\mathbf{A}|$ along the fourth column, yields:

$$t\begin{vmatrix} t & 3 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = t\Big(t\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}\Big) = t\big(-2t-2\big) = \underline{\underline{-2t(t+1)}}.$$

Then, $|\frac{1}{2}\mathbf{AA}'| = (\frac{1}{2})^4|\mathbf{A}|\,|\mathbf{A}| = \frac{1}{16}\,|\mathbf{A}|^2 = \underline{\underline{\frac{1}{4}\,t^2\,(t+1)^2}}$.

2021 corr.: by (c)

(d) Unique solution for all $t \notin \{-1, 0\}$ (by (c)). Remaining cases:

- Case $t = 0$: 4th equation says $0 = 0$ and x_4 does not enter and can be chosen freely provided there is a solution. Considering the equation system in three variables with coefficient matrix $\begin{pmatrix} t & 3 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, it has determinant equal to $-2(t+1) = -2 \neq 0$, and so there is a unique solution for those three variables, with x_4 free. Infinitely many solutions.
- Case $t = -1$: Eliminating:

$$\left(\begin{array}{cccc|c} -1 & 3 & 1 & 0 & -1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ -1 & 0 & 0 & -1 & -1 \end{array}\right) \begin{array}{l} \\ \leftarrow + \\ \\ \leftarrow + \end{array} \sim \left(\begin{array}{cccc|c} -1 & 3 & 1 & 0 & -1 \\ 0 & 2 & 2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 3 & 1 & -1 & -2 \end{array}\right) \begin{array}{l} \\ \leftarrow + \\ -2 \\ \\ \end{array}$$

and now the second equation says $0 = -3$. No solution.

Problem 2 notes:

(a) *The other matrix products are indeed well-defined, but the question was phrased so that one need not argue for that.*

(b) *It is also OK to point out that $\mathbf{AA}'$ must be symmetric, so $\beta = m_{12} = m_{21} = t - 2$.*

(c) *The $\frac{1}{2}$ was deliberately introduced to check whether they know scalings should be raised to the nth power.*
It is be acceptable for full score to calculate $|\frac{1}{2}\mathbf{M}|$ from the matrix itself, although it is not time-efficient.

(d) *For case $t = 0$, it is a serious error to merely notice that x_4 is free without checking that there is a solution for the other three.*

Also, there are likely those who will go on Gaussian eliminating the whole thing without using any information about determinants. That must also be accepted; but division by zero is not.

Problem 3

(a) Fix $p > 0$. Find constants α, β and γ such that $\displaystyle\int z^{\alpha}(\beta + \gamma \ln z)\, dz = z^p \ln z + C$.

(b) Consider the differential equation $\quad \dot{x} + x\,/\,t = 1 + 2\ln t \qquad$ (valid for > 0).

- Show that the particular solution passing through $(t, x) = (1, 0)$, is $t \ln t$.
- Find the general solution.
  (*Hint:* If you do not want to look up a formula: what is $\frac{1}{t} \cdot \frac{d}{dt}(t\, x(t))$?)

Problem 3 solved:

(a) Differentiating the right-hand side wrt. z, we get $pz^{p-1} \ln z + z^p \cdot 1/z = z^{p-1} \cdot (p \ln z + 1)$. Identifying coefficients, $\underline{\underline{\alpha = p - 1, \quad \beta = 1, \quad \gamma = p}}$.

(b)
- $t \ln t$ is ~~1 for $t == 0$~~ 0 for $t = 1$ (corr. 2020), so we only need to show that it satisfies the differential equation. As $(t \ln t)' = \ln t + 1$, the left-hand side becomes $\ln t + 1 + t \ln t\ /t$ which equals the right-hand side.
- $\frac{1}{t}\,\frac{d}{dt}(tx) = \frac{1}{t}(x + t\dot{x})$ equals the left-hand side of the differential equation and thus the right-hand side, so $\frac{d}{dt}(tx) = t + 2t\ln t$. Integrating, $tx = C + \int t(1 + 2\ln t)dt = C + t^2 \ln t$ using part (a) with $p = 2$. So the general solution is $\underline{\underline{x(t) = t \ln t + C/t}}$.

Problem 3 notes: *In (b) one can also first do the general solution and then show the particular solution asked is $C = 0$. Alternatively, one can use that the linear differential equation has general solution equal to $t \ln t + Cu(t)$ where $Cu(t)$ is the general solution of $\dot{x} + x/t = 0$, which is solved for $tx =$ constant.*

Problem 4 Let $r > 0$ be a constant. Define $g(t) = \frac{e^{-t^2}}{t^r} \cdot \frac{e^{rt} - e^{-rt}}{e^{rt} + e^{-rt}}$ for $t > 0$.

(a)
- Show that $\lim_{t\to 0^+} g(t) = \lim_{t\to +\infty} g(t)$ for all $r \in (0,1)$. (Calculate both!)
- Would the equality hold if $r \geq 1$?

(b) Can we (with Mathematics 2 tools) use the extreme value theorem to prove the fact that for $r \in (0,1)$, then g has some global maximum point t_*?

(c) Change the «2» exponent in g slightly to form the function $h(t) = \frac{e^{-t^{2.019}}}{t^r} \cdot \frac{e^{rt} - e^{-rt}}{e^{rt} + e^{-rt}}$.
Approximately how many *percent* does the maximum *value* change?
(Take for granted that a new maximum exists. Express your answer in terms of t_*.)

Problem 4 notes first: *In (a), use of l'Hôpital's rule must be justified. In particular, the limit as $t \to +\infty$ was intended to catch those who don't do that job. That a limit can shift discontinuously from 0 via 1 to ∞ as an exponent crosses a certain value, was covered in hand-in #4. In (b), a satisfactory answer is given below, although it is actually* possible *to use the extreme value theorem in a proof, relegated to an endnote.*

Problem 4 solved:

(a) When $t \to 0^+$, $\frac{e^{-t^2}}{e^{rt}+e^{-rt}} \to \frac{1}{2} \neq 0$, so $g(t) \to \frac{1}{2}\lim_{t\to 0^+} \frac{e^{rt}-e^{-rt}}{t^r}$ which is «$\frac{0}{0}$». By l'Hôpital:

$$\lim_{t\to 0^+} g(t) = \frac{1}{2}\lim_{t\to 0^+} \frac{e^{rt} - e^{-rt}}{t^r} = \frac{1}{2}\lim_{t\to 0^+} \frac{re^{rt} + re^{-rt}}{rt^{r-1}} = \lim_{t\to 0^+} t^{1-r} = 0.$$

When $t \to +\infty$: Because $e^{\pm rt}e^{-t^2} = e^{(\pm r - t)t}$ and the exponent $\to -\infty$, the numerator tends to zero while the denominator tends to infinity. So this limit is 0 as well.

For the second bullet item: The lim as $t \to +\infty$ does not depend on $r > 0$, but the limit as $t \to 0^+$ would be 1 for $r = 1$, and infinite if $r > 1$. The answer is no.

(b) No [although see the endnote]; in order to apply the extreme value theorem, we need a closed and bounded set. The domain of g is $(0, \infty)$ which isn't closed and bounded (in fact, it is neither).

(c) We change an exponent, call it η, from 2. The derivative wrt. the exponent is

$$\frac{\partial}{\partial\eta}\Big[\frac{e^{-t^\eta}}{t^r} \cdot \frac{e^{rt} - e^{-rt}}{e^{rt} + e^{-rt}}\Big]\Big|_{t=t_*} = \frac{e^{rt_*} - e^{-rt_*}}{e^{rt_*} + e^{-rt_*}} \cdot \frac{e^{-t_*^\eta}}{t_*^r}\frac{\partial}{\partial\eta}\big[-t^\eta\big]\Big|_{t=t_*} = -t_*^\eta g(t_*)\ln t_*$$

Increasing η by 0.019, the relative change is $\approx -0.019t_*^2 \ln t_*$, or $-1.9t_*^2 \ln t_*$ percent.

Problem 5

(a) $(x_1, y_1) = (2, 4)$ is a stationary point for the function $F(x, y) = -ye^{-x} - (x^2 - y)e^{-2}$. Classify this point using the second-derivative test.

Consider now the problems

$$\max\ -ye^{-x} \quad \text{subject to} \quad x^2 - y = 0 \tag{L}$$

$$\max\ -ye^{-x} \quad \text{subject to} \quad x^2 - y \le 0 \quad \text{and} \quad x^4 + y^2 \le 32 \tag{K}$$

(b)
- State the Lagrange conditions associated with problem (L); call the multiplier λ.
- State the Kuhn–Tucker conditions associated with problem (K).

(c) The information given in part (a) tells us that $(x_1, y_1) = (2, 4)$ satisfies the Lagrange conditions associated with problem (L), with multiplier $\lambda = e^{-2}$.
- Does (x_1, y_1) satisfy the Kuhn–Tucker conditions associated with problem (K)?
- Consider problem (K). Is it possible for a point to satisfy the constraints and the Kuhn–Tucker conditions with $\lambda = 0$? Again, λ is the multiplier on $x^2 - y$. (*Hint:* Remember that $y \ge x^2 \ge 0$.)

Problem 5 solved:

(a) $F'_y(x, y) = -e^{-x} + e^{-2}$, so $F''_{yy}(x, y) = 0$, $F''_{xy}(x, y) = e^{-x}$, and the Hessian determinant becomes $0 \cdot F''_{xx}(x, y) - e^{-2x} < 0$ (no matter what F''_{xx} is). Saddle point.

(b) Let $L(x, y) = -ye^{-x} - \lambda(x^2 - y) - \mu(x^4 + y^2 - 32)$; to get the Lagrangian for problem (L), put $\mu = 0$. We will need the partial derivatives $L'_x(x, y) = ye^{-x} - 2\lambda x - 4\mu x^3$ and $L'_y = -e^{-x} + \lambda - 2\mu y$.

- Lagrange conditions:

$$ye^{-x} - 2\lambda x = 0 \qquad -e^{-x} + \lambda = 0 \qquad y = x^2$$

- Kuhn–Tucker conditions:

$$ye^{-x} - 2\lambda x - 4\mu x^3 = 0$$
$$-e^{-x} + \lambda - 2\mu y = 0$$
$$\lambda \ge 0 \quad \text{with } \lambda = 0 \text{ if } y > x^2$$
$$\mu \ge 0 \quad \text{with } \mu = 0 \text{ if } x^4 + y^2 < 32.$$

(c)
- Yes, because $\lambda = e^{-2} > 0$ (OK for an active constraint) and with $\mu = 0$.
- With $\lambda = 0$, the first-order conditions read $ye^{-x} = 4\mu x^3$ and $e^{-x} = -2\mu y$, the latter being impossible since $y \ge 0$ and $\mu \ge 0$.

2021 corrections: Original version had by mistake swapped the "4" and "2" exponents.

(b) fixes: Lagrangian (red); 1st partial derivatives; first-order conditions

(c): FOCs fixed, the conclusions stand.

Problem 5 notes: *In (b), alternative equivalent forms are just as good. They are free to include admissibility in the Kuhn–Tucker conditions as well. In (c) first bullet item, the course does per convention not include admissibility, so one can hardly expect that it is checked (nor was that the purpose of the question).*

Problem 4(b) note (for the more ambitious student)

It is possible to produce an extreme value theorem-based proof, because the limits of part (a) are a common value ℓ (an essential assumption; otherwise, a strictly monotone function would be a counterexample! But ℓ could be infinite, so this argument applies to Problem 4 (b)).

The function g satisfies $g(t) > 0 = \ell$. For each $\epsilon > 0$, there is an $M \in (0, 1)$ such that $0 < g(t) < \epsilon$ when $t < M$ or $t > 1/M$. Since g is continuous and the extreme value theorem grants an extremum over the interval $[M, 1/M]$. Now, let $\epsilon > 0$ be so small that $g(t)$ is *somewhere* $\geq \epsilon$ (such an ϵ exists, since $g > 0$). Then the t_* which maximizes g over $[M, 1/M]$ is a global maximum; $g(t_*) \geq g(t)$ for all t in the interval, and $> \epsilon > g(t)$ for all t not in the interval, thus $\geq g(t)$ for all t, the definition of a global max.

The assumption that $g(t) > \ell = 0$ simplifies the writing, but is not essential; the curious mind can try to complete the proof by just assuming g continuous and $\ell = \lim_{t \to 0^+} g(t) = \lim_{t \to +\infty} g(t)$ (possibly infinite).