University of Oslo / Department of Economics / NCF

ECON3120/4120 Mathematics 2: the 2020-12-03 exam solved

This document solves the exam and gives guidelines for the grading process. Updated post grading with information for the appeals committee.

- *Standard disclaimer:*
  - This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
  - The document reflects what was expected in that particular semester, and which may not be applicable to future semesters.
- *Weighting:* Suggestions were stated in the problem set, rather than suggesting a usual (for this course) uniform over letters. The committee can deviate at their discretion. The appeals committee can deviate at their discretion.

The document will restate each problem as given, each followed by a solution/annotations. Generally, what is in sans serif font in what is otherwise a solution, is a comment / an annotation; what says «Notes» in paragraph headings is of course notes as well.

Special considerations (I) for the 2020 exam: the format and the problem set

- The format is exceptional to this course: 5 hrs with more tools available.
- The committee must exercise considerable discretion. There is a risk that the format and the set considered together, miss out on the usual level of difficulty, and this exam cannot make a claim to suit specific grading thresholds[1]. Therefore it is suggested that the committee attach more than usual weight to the official grade descriptions, and also consider the empirical grade distribution.[2]
- **This item updated post grading:** The «pre-2019 defaults» turned out surprisingly good with only minor adjustments: there were very few papers near 88 points, so slacking the requirement for «A» a couple percentage points produced a clear distinction between A and B; a good distinction between D and E was achieved by raising the bar for D a point or two; and, there were very few near-passing fails.

  The appeals committee must still consider to compare to a representative sample of the exam papers submitted. One cannot expect/trust this note to guide the quantification of grades for individual papers or a low number of such.

[1] which up to 2018 defaulted to 91/75/55/45/40 percent in this course; the most recent four-hour Mathematics 2 exam with a changed format did invoke Matematikkrådet's slightly tougher scale.

[2] From the Department's reports for five years 2015-2019, both course codes merged, the fail rate is 19 % and the distribution over passes is: Starting at A: **7 % + 20 % + 37 % + 21 % + 14 %** (ending at E). That is cumulatively 7, 27, 64, 86, 100. Raw numbers per course code:
3120: 11+30+42+20+15 of 118 passed, and additionally 33 fails.
4120: 23+66+136+81+54 of 360 passed, and additionally 79 fails.

Special considerations (II) for the 2020 exam: the submissions, and how to and resolve submission-related technical issues

- The «one PDF for each of problems 1–5» was intended to reduce the number of upload issues. It is not the intention to penalize candidates who submit a wrong scan to the wrong number. The committee must however take note if there are multiple uploads which differ. The administration can find upload timestamp.
- Some candidates needed to rely on the in-case-of-emergencyies e-mail address. These (partial) submissions are attached to the Inspera upload. When such an attachment makes for conflicting versions of a particular answer, it is *most likely* that the attachment shall be taken to supersede the Inspera upload, but the committee must exercise judgement and could consult the Department for further technical information.

A correction during the exam: Problem 1, as noted in the margin. Graders should discuss whether this could have had any impact – given the answers, and the fact that the issue was only reported to teacher as late as 45 minutes before deadline despite concerning problem 1. The announcement was made in Canvas 13:46.

Problem 1 of 5. ***Suggested weight: 15 percent*** Take for granted that the equation system

$$4x^a + y^2x^{3/4} - xp/a = 0$$
$$(1-4a)x^{7/4}y + ax^{3/4} - yw = 0$$

determines continuously differentiable functions $x = x(a, p, q))$ and $y = y(a, p, w)$ around the point where $x = y = p = 1$, $a = 1/5$, $w = 2/5$.

Typo: should be $x(a, p, q)$

(a) Differentiate the system.
(Possible hint: certain terms may benefit from so-called logarithmic differentiation.)

(b) Calculate $\dfrac{\partial x}{\partial a}(\frac{1}{5}, 1, \frac{2}{5})$.

How to solve:

(a) Differentiating the system yields

$$4ax^{a-1}\,dx + (4x^a \ln x)\,da + 2yx^{3/4}\,dy + \frac{3}{4}y^2x^{-1/4}\,dx - \frac{p}{a}\,dx - \frac{x}{a}\,dp + \frac{xp}{a^2}\,da = 0 \qquad \text{and}$$
$$-4x^{7/4}y\,da + (1-4a)\frac{7}{4}x^{3/4}y\,dx + (1-4a)x^{7/4}dy + x^{3/4}da + \frac{3}{4}ax^{-1/4}dx - w\,dy - y\,dw = 0$$

(It is OK to leave it like this without gathering terms.)

(b) Put $dw = dp = 0$ and insert the point coordinates given:

$$4 \cdot \frac{1}{5}\,dx + 0 + 2\,dy + \frac{3}{4}\,dx - 5\,dx + 25\,da = 0$$
$$-4\,da + (1 - \frac{4}{5})\frac{7}{4}\,dx + (1 - \frac{4}{5})\,dy + da + \frac{3}{20}\,dx - \frac{2}{5}\,dy - 0 = 0$$

To get rid of denominators, scale both equations by 20. Order terms to get

$$-69\,dx + 40\,dy + 500\,da = 0$$
$$-60\,da + 10\,dx - 4\,dy = 0$$

Adding 10 of the latter to the former eliminates dy and yields $(100-69)dx + (500-600)da = 0$ so that the answer is $\underline{\underline{100/31}}$.

Problem 2 of 5. ***Suggested weight: 25 percent*** Throughout this problem, the prime symbol denotes matrix transpose and $\mathbf{I}$ is the 3×3 identity matrix.

For each real q, let $\mathbf{A}_q = \begin{pmatrix} q & 1 & 0 \\ 2 & q & -3 \\ -4 & 0 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Denote by $\mathbf{v}_q = \begin{pmatrix} q \\ 2 \\ -4 \end{pmatrix}$ the 1st column of $\mathbf{A}_q$.

(a)
- Calculate $\mathbf{A}_q\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$ and $\mathbf{A}_q\mathbf{B}$ and $(\mathbf{A}_q - q\mathbf{I})(q\mathbf{B} - \mathbf{I})$ or point out that they are not defined.
- It is a fact that element $(2, 3)$ of $\mathbf{A}_q\mathbf{B}$ equals element $(3, 2)$ of $\mathbf{B}\mathbf{A}'_q$. How to see that from the rules of matrix products, without calculating and comparing?

(b)
- Explain, *without* calculating any cofactors, why the determinant of $\mathbf{A}_0$ is zero.
- Show that there is no nonzero integer q such that the determinant of $\mathbf{A}_q$ equals zero. (In this bullet item you *are allowed to* calculate cofactors.)

(c) For what value(s) – if any! – of q will the equation system $\mathbf{A}_q\mathbf{x} = \mathbf{v}_q$ have no solution, resp. precisely one solution, resp. more than one solution? Here, $\mathbf{x}$ is the unknown.

(d) Pick (your choice!) q as an integer ≥ 5. Invert $\mathbf{A}_q$, for that choice of q.

On the solution

(a) Notes on part (a): The first bullet item is intended to catch those who think matrix multiplication is performed elementwise. Also one can take note that the first question shows that there is a solution to the equation system in part (c). The second bullet item uses the symmetry of $\mathbf{B}$.

- $\mathbf{A}_q\left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$ is the first column of $\mathbf{A}_q$, i.e. $\underline{\underline{= \mathbf{v}_q}}$. (this suffices for this question, but expect answers to look different.).

$$\mathbf{A}_q\mathbf{B} = \begin{pmatrix} 3q + 2 + 0 & 2q & q \\ 3 \cdot 2 + 2q - 3 & 2 \cdot 2 & 2 \\ -4 \cdot 3 + 0 + 6 & -4 \cdot 2 & 4 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 3q + 2 & 2q & q \\ 3 + 2q & 4 & 2 \\ -6 & -8 & -4 \end{pmatrix}}}$$

$$(\mathbf{A}_q - q\mathbf{I})(q\mathbf{B} - \mathbf{I}) = q\mathbf{A}_q\mathbf{B} - q^2\mathbf{B} - \mathbf{A}_q + q\mathbf{I}$$

$$= \begin{pmatrix} 3q^2 + 2q & 2q^2 & q^2 \\ 3q + 2q^2 & 4q & 2q \\ -6q & -8q & -4q \end{pmatrix} - \begin{pmatrix} 3q^2 & 2q^2 & q^2 \\ 2q^2 & 0 & 0 \\ q^2 & 0 & 0 \end{pmatrix} - \begin{pmatrix} q & 1 & 0 \\ 2 & q & -3 \\ -4 & 0 & 6 \end{pmatrix} + \begin{pmatrix} q & 0 & 0 \\ 0 & q & 0 \\ 0 & 0 & q \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 2q & -1 & 0 \\ 3q - 2 & 4q & 2q + 3 \\ -q^2 - 6q + 4 & -8q & -3q - 6 \end{pmatrix}}}$$

- Element $(2,3)$ of $\mathbf{A}_q\mathbf{B}$ equals element $(3,2)$ of $(\mathbf{A}_q\mathbf{B})'$, which equals $\mathbf{B}'\mathbf{A}_q'$, and $\mathbf{B}' = \mathbf{B}$.

(b)
- If $q = 0$ the last two rows are proportional.
  (Equally good is to point out that the first and the last column are proportional.)
- Cofactor expansion along the first row yields $q\begin{vmatrix} q & -3 \\ 0 & 6 \end{vmatrix} - 1\begin{vmatrix} 2 & -3 \\ 4 & 6 \end{vmatrix} + 0 = 6q^2 - 0$ is zero only if $q = 0$.

(c) By (b), there is $\underline{\underline{\text{precisely one solution if and only if } q \neq 0}}$. It remains to check whether $q = 0$ yields solution at all (if so, there are infinitely many); from (a), one solution is $(1,0,0)'$ and so there are $\underline{\underline{\text{infinitely many solutions when } q = 0.}}$

(d) Let $q = 5$. We already have two cofactors, so we calculate the full matrix:

$$\mathbf{K} = \begin{pmatrix} 6q & 0 & 0-(-4q) \\ -(6-0) & 6q-0 & -(0-(-4)) \\ -3-0 & -(-3q-0) & q^2-2 \end{pmatrix} \overset{(q=5)}{=} \begin{pmatrix} 30 & 0 & 20 \\ -6 & 30 & -4 \\ -3 & 15 & 23 \end{pmatrix}$$

$$\text{and so} \qquad \mathbf{A}_5^{-1} = \frac{1}{6\cdot 5^2}\mathbf{K}_5' = \underline{\underline{\frac{1}{150}\begin{pmatrix} 30 & -6 & -3 \\ 0 & 30 & 15 \\ 20 & -4 & 23 \end{pmatrix}}}$$

Note: The example used $q = 5$ because smaller numbers would tend to be the default. Maybe 8 is easier for Gaussian elimination. The adjugate isn't that hard for general q either, as seen above, so the inverse would for every $q \neq 0$ turn out as $\frac{1}{6q^2}\begin{pmatrix} 6q & -6 & -3 \\ 0 & 6q & 3q \\ 4q & -4 & q^2-2 \end{pmatrix}$.

Problem 3 of 5. ***Suggested weight: 20 percent***

(a) In this part, you are allowed to manipulate functions before and after antidifferentiating, but there is no score for differentiating the right-hand side.

- Use the substitution $u = 1 - e^{-x}$ to show that $\displaystyle\int (e^x - 1)^{-1} dx = C_1 + \ln\left|1 - e^{-x}\right|$
- Show *by antidifferentiation* that there exists a base number B (constant!) such that $\displaystyle\int \log_B(y^3)\, dy = C_2 + 36y \ln\frac{y}{e}$.

For part (b), let B be the number as in part (a) – note that you are not required to insert for it, you can just write it as "B".

(b) Consider the differential equation $\dot{x} = (e^x - 1)\log_B(t^3)$.
Find the following two particular solutions:

- The one for which $x(e) = 1$.
- The one for which $x(e) = 0$.

Notes: Part (a) requires a bit of manipulating exp and log, and that is intentionally part of the question. Part (b): The lectures and hand-ins have repeatedly urged to check for zero before dividing – with «separating» as formal division used as an example on how to «lose» solutions – that I found it only fair to reward those who paid attention.

Solution:

(a)
- $u = 1 - e^{-x}$ yields $du = e^{-x}dx$ and $\frac{1}{e^x-1}dx = \frac{e^{-x}dx}{1-e^{-x}} = \frac{du}{u}$. Integrating, $\int \frac{1}{u}du = C_1 + \ln|u| = C_1 + \ln|1 - e^{-x}|$.
- $\log_B(y^3) = \frac{3}{\ln B}\ln y$ and $\ln y$ has an antiderivative $y\ln y - y$. The right-hand side can be written $C_2 + 36y(\ln y - \ln e) = C_2 + 36y(\ln y - 1)$, so it is OK with B such that $\frac{3}{\ln B} = 36$ (i.e. $B = e^{1/12}$).

(b) Separable differential equation with constant solution where $e^x - 1 = 0$, i.e. $x \equiv 0$. Therefore, $x(t) \equiv 0$ is the answer to the second bullet item.

For the other (i.e. first bullet item) particular solution, we separate into $\frac{dx}{e^x-1} = \log_B(t^3)\, dt$, and integrate using (a): $\ln|1 - e^{-x}| = C + 36t\ln(t/e)$ with C determined by $\ln|1 - e^{-1}| = C + 36\ln(e/e)$ and $\ln(e/e) = 0$. We exponentiate:

$$|1 - e^{-x}| = (1 - e^{-1}) \cdot e^{36t\ln(t/e)} =$$

and when $x = 1$, $|1 - e^{-x}| = 1 - e^{-x}$ so we can remove the absolute value bars and get $e^{-x} = 1 - (1 - e^{-1})(t/e)^{36t}$. Apply ln and flip sign:

$$\underline{\underline{x(t) = -\ln\left(1 - (1 - e^{-1})(t/e)^{36t}\right)}}$$

Problem 4 of 5. ***Suggested weight: 25 percent*** For each constant $a \in (0,1)$, define the function $f(K,L) = \sqrt{K^a L^{1-a}} + a \cdot (KL)^{1/4}$.

(a) It is a fact that the sum of concave functions is concave. Show that f is concave. You are allowed to use known properties of the most well-known production functions, as long as you point out that you are in the applicable parameter range.

Consider from now on the maximization problem

$$\max f(K,L) \qquad \text{subject to} \qquad K + L \leq 2, \quad K \geq a, \quad L \geq a \tag{P}$$

(b) State the Kuhn–Tucker conditions associated with problem (P).

(c) Without actually solving, do we know enough to tell:

- whether any point $(\hat{K}, \hat{L})$ satisfying the Kuhn–Tucker conditions – provided it exists! – must be optimal?
- whether any such point $(\hat{K}, \hat{L})$ satisfying the Kuhn–Tucker conditions does exist?

(You can take for granted that if a maximum exists, the Kuhn–Tucker conditions *must* hold there.)

(d) When $a = 1/2$, you can take for granted that the point $(1,1)$ is optimal (and no other point is optimal). Approximately how much does would optimal value change if a were reduced to 0.48?

Notes first: (a) The question is formulated that way because some students would likely take as known that Cobb–Douglas are concave with the right parameters. It was intended to save others the time they might have taken to calculate second derivatives. (b) ff.: Because the literature is a bit ambiguous as to whether the condition set with the «Kuhn–Tucker» name includes admissibility, one must give a bit of leeway as long as the students are doing the right thing. (c) Constraint qualifications are not on curriculum, so the conditions are taken as necessary without any mention of neither CQ or Slater. The extreme value theorem has arguably been stressed more than sufficiency in this course (because it isn't obvious to all students that the extreme value theorem applies in a single variable too, and therefore a few of those problems were given as well). (d): Application of the full envelope theorem – where the parameter enters both in the objective and the constraints – might be the hardest question on this exam.

Solution:

(a) $\sqrt{K^a L^{1-a}} = K^{a/2} L^{(1-a)/2}$ is Cobb–Douglas, sum of exponents $= 1/2$, and is concave. The same is the case for $a \cdot (KL)^{1/4} = aK^{1/4}L^{1/4}$.

(b) Let the Lagrangian be $F(K,L) = K^{a/2}L^{(1-a)/2} + aK^{1/4}L^{1/4} - \lambda(K+L-2) - \gamma(a-K) - \beta(a-L)$. The Kuhn–Tucker conditions read:

$$0 = \frac{a}{2}K^{a/2-1}L^{(1-a)/2} + \frac{1}{4}K^{-3/4}L^{1/4} - \lambda + \gamma \tag{1}$$

$$0 = \frac{1-a}{2}K^{a/2}L^{-a/2-1/2} + \frac{a}{4}K^{1/4}L^{-3/4} - \lambda + \beta \tag{2}$$

$$\lambda \geq 0 \qquad \text{with} = 0 \text{ if } K + L < 2 \tag{3}$$

$$\gamma \geq 0 \qquad \text{with } \gamma = 0 \text{ if } K > a \tag{4}$$

$$\beta \geq 0 \qquad \text{with } \beta = 0 \text{ if } L > a \tag{5}$$

(or equivalent formulations. E.g. the latter can be written as $\beta = 0 = \beta(L-a)$.)

(c) Affirmative for both questions: The admissible set is a closed triangle, so it is bounded, and the function is defined and continuous there. So the extreme value theorem grants that a maximum exists, and Kuhn–Tucker conditions must hold there. Conversely, if the conditions hold at some admissible point, the concavity of the Lagrangian implies that it solves the problem.
(Note: it is concavity *of the Lagrangian,* which follows from *linearity of the functions $K+L-2$, $a-K$ and $a-L$* together with the concavity of f.)

(d) The envelope theorem. Using logarithmic differentiation, we have:

$$\frac{\partial}{\partial a}\Big[K^{a/2}L^{(1-a)/2} + aK^{1/4}L^{1/4} - \lambda(K+L-2) - \gamma(a-K) - \beta(a-L)\Big]$$
$$= K^{a/2}L^{(1-a)/2}\frac{\partial}{\partial a}\Big[\frac{a}{2}\ln K + \frac{1-a}{2}\ln L\Big] + K^{1/4}L^{1/4} - \gamma - \beta$$

which when we insert $K = L = 1$ reduces to $1 \cdot [\frac{1}{2}\ln 1 - \frac{1}{2}\ln 1] + 1 - \gamma - \beta$, and $\gamma = \beta = 0$ because $K = 1 > a$ and $L = 1 > a$. So the change is $\approx (0.48 - \frac{1}{2}) \cdot 1 = -0.02$.
(Note that one *can not pretend the constraints don't exist!* A correct answer needs «both $1 - \gamma - \beta$ and applying the Kuhn–Tucker conditions to find $\gamma = \beta = 0$».
There is room for simplification by inserting $(K,L) = (1,1)$ *first,* as we are not asked for a *general* expression for the derivative wrt. a; we only need the partial change in a from the given $(K, L, \lambda, \gamma, \beta)$. We get $\frac{\partial}{\partial a}\big[1 + a\cdot 1 - \lambda(2-2) - \gamma(a-K) - \beta(a-L)\big] = 1 - \gamma - \beta = 1$ again. This approach is «dangerous» unless you know precisely what you are doing.)

Problem 5 of 5. ***Suggested weight: 15 percent*** For each constant $r > 0$, let the function $g(x) = \dfrac{4x}{1+x^2}\ln(e + rx^2) - (1 + x^2)e^{-rx}$ be defined for all real x.

(a) Determine $\lim_{x\to-\infty} g(x)$ and show that $\lim_{x\to+\infty} g(x) = 0$

(b) Show that g has a zero $x_0 \in (0, 1)$.

Notes:

- It was deliberate that the last-in-the-set question in the problem set would only need one sheet to be scanned and uploaded in the zeitnot. (Update: I was informed that this did not materialize and that several papers spent multiple pages on this.)
- The reason for asking for both limits in (a), was to easier be able to identify those who just claim that the exponential will win. That is also the reason for the log term – which is ln([nonlinear]) to call for more thought.
- Of course there are fancy ways to handle the terms here: the first can be written as $4\frac{x^2}{1+x^2} \cdot \frac{\ln(e+rx^2)}{x}$ where $\frac{x^2}{1+x^2} = \frac{1}{1+x^{-2}} \to 1$. Also the $x \to -\infty$ limit follows easily if one observes that for $x < 0$, $g(x) = [\text{something negative}] - (1+x^2)e^{-rx} < -(1 + x^2)e^{-rx}$ which $\to -(+\infty)\cdot(+\infty)$. The below proposed solution does not aim at clever shortcuts.
- Upon applying l'Hôpital's rule, one must check that it does indeed apply.

Solution:

(a) For both limits, the term $\lim \frac{4x}{1+x^2}\ln(e + rx^2)$ is (plus/minus) «$\frac{\infty}{\infty}$», and we can apply l'Hôpital to get $4\lim \frac{\ln(e+rx^2)+\frac{x\cdot 2rx}{e+rx^2}}{2x}$. The term $\frac{\ln(e+rx^2)+\frac{x\cdot 2rx}{e+rx^2}}{2x}$ equals $1\frac{r}{e/x+rx}$ which $\to 0$, while $2\lim\frac{\ln(e+rx^2)}{x}$ is again (plus/minus) «$\frac{\infty}{\infty}$» and equals $2\lim\frac{2rx}{e+rx^2}$ which as we just saw is 0. Therefore:

$\lim_{x\to-\infty} g(x) = 0 - \lim_{x\to-\infty}(1 + x^2)e^{-rx} = -\infty\cdot\infty = \underline{\underline{-\infty}}$;

$\lim_{x\to+\infty} g(x) = 0 - \lim_{x\to+\infty}\dfrac{1+x^2}{e^{rx}}$ is $\frac{\text{[polynomial growth]}}{\text{[exp growth]}}$ and therefore also zero as should be shown. (Also confirmed by l'Hôpital twice.)

(b) $g(0) = 0 - 1\cdot e^0 = -1 < 0$ while $g(1) = \frac{4}{2}\ln(e + r) - 2e^{-r} = 2\cdot[\ln(e + r) - e^{-r}]$ is > 0 because $\ln(e + r) > \ln e = 1 > e^{-r}$. The intermediate value theorem grants a zero in $(0, 1)$.