i Candidate instructions

ECON3120/4120 – **Mathematics 2: Calculus and Linear Algebra** – Open book, home exam

Exam date and time: Thursday, 3 December, 2020 from 09.00 – 14.00 (five hours)

Language: The examination text is given in English. You may submit your response in Norwegian, Swedish, Danish or English.

Guidelines: You should upload your text in pdf format - **one pdf per problem.** You can scroll back and forth in the problem set.

You should familiarize yourself with the rules that apply to the use of sources and citations.

The exam lasts for only five hours. We recommend that you use the available time to work on the problem set, as well as allocate time to scan attachments with graphs and/or equations.
The problem set: The problem set consists of five problems, with several sub-problems. They count as indicated.

Note: You can resize the question by clicking on the three dots on the right, hold and pull to the right. Similarly for the three dots at the bottom, click, hold and pull down. Then the text will be larger. Or you can download each problem from the link (recommended).

Digital hand drawings/graphs/equations: You will find information about options for hand drawings on this website: https://www.uio.no/english/studies/examinations/submissions/options-for-hand-drawings.html

Submission in Inspera

- Read more about exam and submission in Inspera. https://www.uio.no/english/studies/examinations/submissions/.
- Remember: It is your responsibility to upload the **correct version of the correct answer**.
- When your answer is uploaded, you will see that the exam is uploaded and saved.
- To submit your answer, please see https://www.uio.no/english/studies/examinations/submissions/submit_answer/. You can either choose the “submit now” or the “Automatic submission”.
- You can make changes in your exam until the deadline.
- You will find the answer under Archives (Check that this is the right answer).

Do you need technical support, or do you have any questions during the exam?

Please send an e-mail, titled “ECON3120/4120” to hjemmeeksamen@sv.uio.no from your university email.

Grading: The grades given: A-F, with A as the best and E as the weakest passing grade. F is fail.
Grades are given: 21 December 2020

ECON4120 Mathematics 2 exam 2020-12-03

- You are required to state reasons for all your answers. For this 2020 exam:
  - Except where otherwise stated, all answers must be justified with calculations as if the exam were a school exam with textbook only;
  - As long as you are invoking theory known from the syllabus, you can reference it as it would have been expected in a school exam; and, where are asked to state particular instances of conditions or other theory, no citation is needed. This item relaxes the rules for referencing and citations; note however that *quoting from a source* is subject to normal citation practice (with quotations marks and references to the source).
    You are not expected to create any bibliography.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.
- "Suggested" weights: the grading committee is free to deviate.

Problem 1 of 5. ***Suggested weight: 15 percent*** Take for granted that the equation system

$$4x^a + y^2x^{3/4} - xp/a = 0$$
$$(1-4a)x^{7/4}y + ax^{3/4} - yw = 0$$

determines continuously differentiable functions $x = x(a, p, q))$ and $y = y(a, p, w)$ around the point where $x = y = p = 1, \quad a = 1/5, \quad w = 2/5$.

(a) Differentiate the system.
(Possible hint: certain terms may benefit from so-called logarithmic differentiation.)

(b) Calculate $\dfrac{\partial x}{\partial a}(\frac{1}{5}, 1, \frac{2}{5})$.

Problem 2 of 5. ***Suggested weight: 25 percent*** Throughout this problem, the prime symbol denotes matrix transpose and $\mathbf{I}$ is the 3×3 identity matrix.

For each real q, let $\mathbf{A}_q = \begin{pmatrix} q & 1 & 0 \\ 2 & q & -3 \\ -4 & 0 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Denote by $\mathbf{v}_q = \begin{pmatrix} q \\ 2 \\ -4 \end{pmatrix}$ the 1st column of $\mathbf{A}_q$.

(a)
- Calculate $\mathbf{A}_q \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$ and $\mathbf{A}_q\mathbf{B}$ and $(\mathbf{A}_q - q\mathbf{I})(q\mathbf{B} - \mathbf{I})$ or point out that they are not defined.
- It is a fact that element $(2,3)$ of $\mathbf{A}_q\mathbf{B}$ equals element $(3,2)$ of $\mathbf{B}\mathbf{A}_q'$. How to see that from the rules of matrix products, without calculating and comparing?

(b)
- Explain, *without* calculating any cofactors, why the determinant of $\mathbf{A}_0$ is zero.
- Show that there is no nonzero integer q such that the determinant of $\mathbf{A}_q$ equals zero. (In this bullet item you *are allowed to* calculate cofactors.)

(c) For what value(s) – if any! – of q will the equation system $\mathbf{A}_q\mathbf{x} = \mathbf{v}_q$ have no solution, resp. precisely one solution, resp. more than one solution? Here, $\mathbf{x}$ is the unknown.

(d) Pick (your choice!) q as an integer ≥ 5. Invert $\mathbf{A}_q$, for that choice of q.

Problem 3 of 5. ***Suggested weight: 20 percent***

(a) In this part, you are allowed to manipulate functions before and after antidifferentiating, but there is no score for differentiating the right-hand side.

- Use the substitution $u = 1 - e^{-x}$ to show that $\displaystyle\int (e^x - 1)^{-1} dx = C_1 + \ln\left|1 - e^{-x}\right|$
- Show *by antidifferentiation* that there exists a base number B (constant!) such that $\displaystyle\int \log_B(y^3)\, dy = C_2 + 36y \ln\frac{y}{e}$.

For part (b), let B be the number as in part (a) – note that you are not required to insert for it, you can just write it as "B".

(b) Consider the differential equation $\dot{x} = (e^x - 1)\log_B(t^3)$.
Find the following two particular solutions:

- The one for which $x(e) = 1$.
- The one for which $x(e) = 0$.

Problem 4 of 5. ***Suggested weight: 25 percent*** For each constant $a \in (0,1)$, define the function $f(K,L) = \sqrt{K^a L^{1-a}} + a \cdot (KL)^{1/4}$.

(a) It is a fact that the sum of concave functions is concave. Show that f is concave. You are allowed to use known properties of the most well-known production functions, as long as you point out that you are in the applicable parameter range.

Consider from now on the maximization problem

$$\max f(K,L) \qquad \text{subject to} \qquad K+L \le 2, \quad K \ge a, \quad L \ge a \tag{P}$$

(b) State the Kuhn–Tucker conditions associated with problem (P).

(c) Without actually solving, do we know enough to tell:

- whether any point $(\hat{K}, \hat{L})$ satisfying the Kuhn–Tucker conditions – provided it exists! – must be optimal?
- whether any such point $(\hat{K}, \hat{L})$ satisfying the Kuhn–Tucker conditions does exist?

(You can take for granted that if a maximum exists, the Kuhn–Tucker conditions *must* hold there.)

(d) When $a = 1/2$, you can take for granted that the point $(1,1)$ is optimal (and no other point is optimal). Approximately how much does would optimal value change if a were reduced to 0.48?

Problem 5 of 5. ***Suggested weight: 15 percent*** For each constant $r > 0$, let the function $g(x) = \dfrac{4x}{1+x^2}\ln(e + rx^2) - (1+x^2)e^{-rx}$ be defined for all real x.

(a) Determine $\lim\limits_{x\to-\infty} g(x)$ and show that $\lim\limits_{x\to+\infty} g(x) = 0$

(b) Show that g has a zero $x_0 \in (0, 1)$.

`<-- end of problem set -->`