# ECON4120 Mathematics 2 postponed exam 2021-01-14

- You are required to state reasons for all your answers. For this 2020 exam:
  - Except where otherwise stated, all answers must be justified with calculations as if the exam were a school exam with textbook only;
  - As long as you are invoking theory known from the syllabus, you can reference it as it would have been expected in a school exam; and, where are asked to state particular instances of conditions or other theory, no citation is needed. This item relaxes the rules for referencing and citations; note however that *quoting from a source* is subject to normal citation practice (with quotations marks and references to the source).
    You are not expected to create any bibliography.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.
- "Suggested" weights: the grading committee is free to deviate.

In this set you will encounter the following concave functions, defined for $x \geq 0$, $y \geq 0$:

$$f(x,y) = x^{2\theta} + y^{2\theta}, \qquad g(x,y) = x^{\theta\cdot(1-\theta)}y^{\theta\cdot(1+\theta)}, \qquad h(x,y) = f(x,y) + g(x,y)$$

where $\theta$ is a constant, always assumed to be in the interval $(0, 1/2)$.

**Problem 1 of 5.** ***Suggested weight: 20 percent*** Let $h(x,y)$ be as given above, and consider the maximization problem $\max_{(x,y)} \big(h(x,y) - x - wy\big)$. The first-order conditions are equivalent to the following system, which you can take for granted will determine $x = x(\theta, w)$ and $y = y(\theta, w)$:

$$\begin{aligned} 2x^{2\theta} + (1-\theta)x^{\theta-\theta^2}y^{\theta+\theta^2} &= x/\theta \\ 2y^{2\theta} + (1+\theta)x^{\theta-\theta^2}y^{\theta+\theta^2} &= wy/\theta \end{aligned} \tag{S}$$

(a) Differentiate the equation system (S).
(*Hint:* it may be useful that $dR = R\, d(\ln R)$ for $R > 0$.)

(b) It turns out that when $w = 11/13$ and $\theta = 1/4$, then $x = y = (13/16)^2 = 169/256$. If $w$ increases to $12/13$ and $\theta$ is kept constant, approximately how much does $x$ change?

(c) If $w$ increases to $12/13$ and $\theta$ is kept constant, approximately how much does the optimal *value* of the maximization problem change?

**Problem 2 of 5.** ***Suggested weight: 20 percent*** Let $a, b, c, d, p, q, r, s$ be constants.

Let $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{M} = \begin{pmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & p & q \\ 0 & 0 & 0 & r & s \end{pmatrix}$. You are free to write $\mathbf{M}$ as $\mathbf{M} = \begin{pmatrix} \mathbf{A} & {}^{0}_{0} & \mathbf{0} \\ 0\ 0 & 5 & 0\ 0 \\ \mathbf{0} & {}^{0}_{0} & \mathbf{P} \end{pmatrix}$

in terms of «blocks» $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and the $2 \times 2$ null matrix.

(a) Calculate the following or point out that the respective matrix product is not defined:

- $\mathbf{Mv}$ and $\mathbf{M}^2\mathbf{v}$ and $\mathbf{Mv}^2$
- element $(4, 5)$ of $(\mathbf{M} - 5\mathbf{I}^{2021})\mathbf{M}$ where $\mathbf{I}$ is the $5 \times 5$ identity matrix.

(b)

- Show that $|\mathbf{M}| = 5\big|\mathbf{PA}\big|$.
- Let $D_i$ be the determinant you get if you replace column $i$ of $\mathbf{M}$ by $\mathbf{v}$. It turns out that several of the $D_i$ will be zero; is that possible to see *without* doing any cofactor expansion, using tools from this course, or would you have to calculate cofactors?

(c) Consider the equation system $\mathbf{Mx} = \mathbf{v}$ for the unknown $\mathbf{x}$.

- What expression for solution does Cramér's rule give you? (You are not allowed to use any other solution method!)
- Show that there will be several solutions if *and only if* the matrix $\mathbf{PAA'A^3P'}$ fails to have an inverse. (The prime symbol denotes transposition.)

**Problem 3 of 5.** ***Suggested weight: 15 percent***

(a)
- *Use the substitution* $u = \ln t$ to calculate $\int \ln t \, dt$.

  This particular substitution is mandatory, and you are not allowed to use any other substitution until after you are done antidifferentiating, only then are you allowed to substitute back.

  You *are* however allowed to use integration by parts after you have substituted.

- Calculate $\int_1^{e^K} (\ln t)^{1+K} \, dt$ for some constant $K > 0$ of your choice.

(b) Consider the differential equation $2\dot{x} + 3 = 4x + 6t$.
- There is a particular solution of the form $x(t) = Qt$. Find the constant $Q$.
- Find the general solution.

**Problem 4 of 5.** ***Suggested weight: 25 percent*** The following Problem (P) is a model for a Pareto efficient allocation of two goods in unit supply between two agents with the same utility function:

$$\max_{x,y} h(x,y) \qquad \text{subject to} \quad \begin{cases} r(x,y) \geq 3 - C \\ 0 \leq x \leq 1, \qquad\qquad 0 \leq y \leq 1 \end{cases} \tag{P}$$

where $r(x,y) = h(1-x, 1-y)$, where $h(x,y) = x^{2\theta} + y^{2\theta} + x^{\theta\cdot(1-\theta)}y^{\theta\cdot(1+\theta)}$ as on page 1, and the constant $C$ is $\in [0,3]$.

(a) It is a fact that the admissible set is nonempty: since $r(0,0) = h(1,1) = 3$, the point $(x,y) = (0,0)$ satisfies the constraints.
Check the other conditions of the extreme value theorem.

From now on, take for granted – whether or not the extreme value theorem applies! – that there exist optimal $x = \phi(C)$ and $y = \psi(C)$.
Take also for granted that $\phi$ and $\psi$ are *continuous* functions of $C$.

(b) Show that there exists a $C \in (0,3)$ such that the optimal value $h(\phi(C), \psi(C)) = \sqrt{2}$.
(*Hint:* Let $V(C) = h(\phi(C), \psi(C))$. Then $V(3) = h(1,1) = 3$ because if $C = 3$ we choose $x = \phi(3) = 1$ and $y = \psi(3) = 1$, allocating everything to one agent.)

(c) True or false? *«For the Kuhn–Tucker conditions associated to problem (P) to hold true, at least two multipliers must be zero.*

(d) Does the point $(x,y) = (\frac{1}{2}, \frac{1}{2})$ satisfy the Kuhn–Tucker conditions?
(The answer may depend on $C$.)

**Problem 5 of 5.** ***Suggested weight: 20 percent*** Let again $\theta \in (0, 1/2)$ and consider the same functions as on page 1, defined (for positive $x$ and $y$) by

$$f(x,y) = x^{2\theta} + y^{2\theta}, \qquad g(x,y) = x^{\theta\cdot(1-\theta)}y^{\theta\cdot(1+\theta)}, \qquad h(x,y) = f(x,y) + g(x,y)$$

(a) What – if anything – does l'Hôpital's rule tell us about $\displaystyle\lim_{\theta\to 0^+} \frac{x^{2\theta} + y^{2\theta} - 2}{x^{\theta\cdot(1-\theta)}y^{\theta\cdot(1+\theta)} - 1}$ ?

(b) It is possible to show that $El_x h(x,y) + El_y h(x,y)$ is a constant, *without* calculating any of these elasticities or any derivatives. How?

(c) Calculate the elasticity of substitution for $f$.

`<-- end of problem set -->`