University of Oslo / Department of Economics / NCF

ECON3120/4120 Mathematics 2: the 2021-01-14 exam

This page: post (ordinary) grading updates.

This problem set turned out unfortunate to say the least, and the grading thresholds had to be tweaked like possibly never before. It did however distinguish adequately between the better and the worse, producing nearly the same ranking of papers among the graders (rank correlation .96.)

After the means employed to each problem as described below, the following grade assignments distinguished the grades, and quite clearly so except the D/E threshold; that is not uncommon, as the E interval is usually narrow.

A: B or A on the default scale

B: C on the default scale

C: D or E on the default scale (actual score interval $[42, 51]$ after rounding up)

D: mid-thirties (interval $[34, 37]$ after rounding up)

E: low thirties ($[30, 33]$ rounding up)

F: twenties and below.

As for the individual problems:

1: Problem 1 (a) and especially (b) turned out too much work, and (b) was arguably harder with numerical values for x and y rather than putting $y = x$ as a letter. We awarded full or near-full score to those who did the right thing without making errors that would zero out too much; close to half the papers could not tell variables from constants (the problem set says $x = x(\theta, w)$ and $y = y(\theta, w)$) and still got scores like $1/3$ or a bit more. The average score on (c) was far from passing.

2: Parts (a) and (b) averaged not far from eighty percent score. 2(c) on the other hand, the connection between uniqueness of solution and nonzero determinant, scored below 30.

3: Although the integrals are not hard and one question in (b) could be answered by just inserting without even solving a differential equation, problem 3 averaged in the forties.

4: Disaster. 4(d) – a fifty–fifty allocation in an Edgeworth box between two agents of the same preferences – averaged worst in the entire set. Even the best-scoring part of 4 – that is (a) – was less than impressive. Candidates who speak about closed function and continuous set have obviously done nothing to check.

5: All three parts averaged twenty to twenty-five percent. Half the candidates answered that l'Hopital's rule applies, and got no further. That does not include those who said that the rule is useless because we get $0/0$, or unnecessary because $0/0$ is zero.

Solution with pre-grading drafted annotations

This document solves the exam and gives guidelines for the grading process.

- *Standard disclaimer:*
  - This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
  - The document reflects what was expected in that particular semester, and which may not be applicable to future semesters.
- *Weighting:* Suggestions were stated in the problem set, rather than suggesting a usual (for this course) uniform over letters. The committee can deviate at their discretion. The appeals committee can deviate at their discretion.

The document will restate each problem as given, each followed by a solution/annotations. Generally, what is in sans serif font in what is otherwise a solution, is a comment / an annotation; what says «Notes» in paragraph headings is of course notes as well.

Special considerations (I) for the 2020 exam: the format and the problem set

- The format is exceptional to this course: 5 hrs with more tools available.
- The committee must exercise considerable discretion. There is a risk that the format and the set considered together, miss out on the usual level of difficulty, and this exam cannot make a claim to suit specific grading thresholds[1]. Therefore it is suggested that the committee attach more than usual weight to the official grade descriptions, and also consider the empirical grade distribution.[2][3]

Special considerations (II) for the 2020 exam: the submissions, and how to and resolve submission-related technical issues

- The «one PDF for each of problems 1–5» was intended to reduce the number of upload issues. It is not the intention to penalize candidates who submit a wrong scan to the wrong number. The committee must however take note if there are multiple uploads which differ, and possibly check timestamps.

[1] which up to 2018 defaulted to 91/75/55/45/40 percent in this course; the most recent four-hour Mathematics 2 exam with a changed format did invoke Matematikkrådet's slightly tougher scale.

[2] From the Department's reports for five years 2015-2019, both course codes merged, the fail rate is 19 % and the distribution over passes is: Starting at A: **7 % + 20 % + 37 % + 21 % + 14 %** (ending at E). That is cumulatively 7, 27, 64, 86, 100. Raw numbers per course code:
3120: 11+30+42+20+15 of 118 passed, and additionally 33 fails.
4120: 23+66+136+81+54 of 360 passed, and additionally 79 fails.

[3] Addendum after ordinary grading: for the adjustments actually made, see the first page.

- At the ordinary exam, a number of candidates needed to rely on the in-case-of-emergencies e-mail address. These (partial) submissions are attached to the Inspera upload. When such an attachment makes for conflicting versions of a particular answer, it is *most likely* that the attachment shall be taken to supersede the Inspera upload, but the committee must exercise judgement and could consult the Department for further technical information.[4]

Next pages: each problem with solution and annotations

[4] Post grading update: and so it was done, all attachments considered to supersede the relevant part of the Inspera upload.

Problem 1 of 5. ***Suggested weight: 20 percent*** Let $h(x, y)$ be as given above, and consider the maximization problem $\max_{(x,y)} \big(h(x, y) - x - wy\big)$. The first-order conditions are equivalent to the following system, which you can take for granted will determine $x = x(\theta, w)$ and $y = y(\theta, w)$:

$$\begin{aligned} 2x^{2\theta} + (1-\theta)x^{\theta-\theta^2}y^{\theta+\theta^2} &= x/\theta \\ 2y^{2\theta} + (1+\theta)x^{\theta-\theta^2}y^{\theta+\theta^2} &= wy/\theta \end{aligned} \tag{S}$$

(a) Differentiate the equation system (S).
(*Hint:* it may be useful that $dR = R\, d(\ln R)$ for $R > 0$.)

(b) It turns out that when $w = 11/13$ and $\theta = 1/4$, then $x = y = (13/16)^2 = 169/256$. If w increases to $12/13$ and θ is kept constant, approximately how much does x change?

(c) If w increases to $12/13$ and θ is kept constant, approximately how much does the optimal *value* of the maximization problem change?

Notes: It is common to have a problem with a system to (a) differentiate and (b) do something about it. For the latter, the essential is get the *method* right for the right entity (in this case: solve for dx) – not get the numbers right. It is not uncommon to get an "A" worth score with completely wrong numbers in the end. Presumably that goes even more for a take-home exam, where those who can frame formula (1') into a CAS would be at advantage – and that is not what the course intends to test.

Part (c) is a common question in an uncommon place: The envelope theorem should be well-known, but it might not be instantly recognizable when put after questions (a) and (b).

How to solve: (c) can be done without (a) or (b), merely using information given in part (b). To illustrate that, it is solved first here.

(c) By the envelope theorem, the derivative is $\frac{\partial}{\partial w}\big(h(x, y) - x - wy\big) = -y$, so the change is $\approx -\frac{1}{13} \cdot (13/16)^2 = \underline{\underline{-\frac{13}{256}}}$. (Arguably, "how much" can be interpreted as absolute value. Committee should exercise best judgement.)

(a) [There are several ways to differentiate. This one uses the hint term-by-term, but for space and linebreaks, two terms change order with the longest first]
Using the hint on the first equation:

$$(1-\theta)x^{\theta-\theta^2}y^{\theta+\theta^2}\, d\Big[\ln(1-\theta) + (\theta-\theta^2)\ln x + (\theta+\theta^2)\ln y\Big]$$
$$+2x^{2\theta}\, d\Big[\ln 2 + 2\theta \ln x\Big] = \tfrac{x}{\theta}\, d\Big[\ln x - \ln\theta\Big]$$

and differentiating:

$$\begin{aligned}(1-\theta)x^{\theta-\theta^2}y^{\theta+\theta^2}\Big[\tfrac{\theta-\theta^2}{x}dx+\tfrac{\theta+\theta^2}{y}dy+\Big(\tfrac{-1}{1-\theta}+(1-2\theta)\ln x+(1+2\theta)\ln y\Big)d\theta\Big]\\ +4x^{2\theta}\Big[\ln x\,d\theta+\tfrac{\theta}{x}dx\Big]=\tfrac{1}{\theta}dx-\tfrac{x}{\theta^2}d\theta\end{aligned}\tag{1}$$

Similarly, on the second equation:

$$\begin{aligned}(1+\theta)x^{\theta-\theta^2}y^{\theta+\theta^2}\;d\Big[\ln(1+\theta)+(\theta-\theta^2)\ln x+(\theta+\theta^2)\ln y\Big]\\ +2y^{2\theta}\;d\Big[\ln 2+2\theta\ln y\Big]=\tfrac{wy}{\theta}\;d\Big[\ln w+\ln y-\ln\theta\Big]\end{aligned}$$

and differentiating:

$$\begin{aligned}(1+\theta)x^{\theta-\theta^2}y^{\theta+\theta^2}\Big[\tfrac{\theta-\theta^2}{x}dx+\tfrac{\theta+\theta^2}{y}dy+\Big(\tfrac{1}{1+\theta}+(1-2\theta)\ln x+(1+2\theta)\ln y\Big)d\theta\Big]\\ +4y^{2\theta}\Big[\ln y\,d\theta+\tfrac{\theta}{y}dy\Big]=\tfrac{y}{\theta}dw+\tfrac{w}{\theta}dy-\tfrac{yw}{\theta^2}d\theta\end{aligned}\tag{2}$$

The answer is formulae (1) and (2).

(b) Put $d\theta=0$, $\theta=\frac{1}{4}$ and $y=x$, and the differentiated system becomes

$$\tfrac{3}{4}x^{-1/2}\Big[\tfrac{3}{16}dx+\tfrac{5}{16}dy\Big]+x^{-1/2}dx=4\,dx\tag{1'}$$

$$\tfrac{5}{4}x^{-1/2}\Big[\tfrac{3}{16}dx+\tfrac{5}{16}dy\Big]+x^{-1/2}dy=4\big[w\,dy+y\,dw\big]\tag{2'}$$

[Here it is likely easier to use Cramér. Let's not take the shortest route:] Multiply by $4x^{1/2}=4\cdot 13/16=13/4$ and put $w=11/13$ and $dw=1/13$

$$3\Big[\tfrac{3}{16}dx+\tfrac{5}{16}dy\Big]+4\,dx=13\,dx\tag{1''}$$

$$5\Big[\tfrac{3}{16}dx+\tfrac{5}{16}dy\Big]+4\,dy=13\big[\tfrac{11}{13}\,dy+\tfrac{13^2}{16^2}\cdot\tfrac{1}{13}\big]\tag{2''}$$

Now (1'') says $3dx+5dy=16\cdot\frac{13-4}{3}dx=48dx$, which yields $dy=9dx$. Insert into (2''):

$$\begin{aligned}5\frac{3+5\cdot 9}{16}dx+4\cdot 9dx&=11\cdot 9dx+\tfrac{13^2}{16^2}\\ (15+36-99)dx&=\tfrac{13^2}{16^2}\\ dx&=\underline{\underline{-\tfrac{13^2}{48\cdot 16^2}}}\end{aligned}$$

Problem 2 of 5. ***Suggested weight: 20 percent*** Let a, b, c, d, p, q, r, s be constants.

Let $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\mathbf{M} = \begin{pmatrix} a & b & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & p & q \\ 0 & 0 & 0 & r & s \end{pmatrix}$. You are free to write $\mathbf{M}$ as $\mathbf{M} = \begin{pmatrix} \mathbf{A} & {}^{0}_{0} & \mathbf{0} \\ 0\;0 & 5 & 0\;0 \\ \mathbf{0} & {}^{0}_{0} & \mathbf{P} \end{pmatrix}$

in terms of «blocks» $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{P} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and the 2×2 null matrix.

(a) Calculate the following or point out that the respective matrix product is not defined:

- $\mathbf{Mv}$ and $\mathbf{M}^2\mathbf{v}$ and $\mathbf{Mv}^2$
- element $(4, 5)$ of $(\mathbf{M} - 5\mathbf{I}^{2021})\mathbf{M}$ where $\mathbf{I}$ is the 5×5 identity matrix.

(b)

- Show that $|\mathbf{M}| = 5\big|\mathbf{PA}\big|$.
- Let D_i be the determinant you get if you replace column i of $\mathbf{M}$ by $\mathbf{v}$. It turns out that several of the D_i will be zero; is that possible to see *without* doing any cofactor expansion, using tools from this course, or would you have to calculate cofactors?

(c) Consider the equation system $\mathbf{Mx} = \mathbf{v}$ for the unknown $\mathbf{x}$.

- What expression for solution does Cramér's rule give you? (You are not allowed to use any other solution method!)
- Show that there will be several solutions if *and only if* the matrix $\mathbf{PAA'A^3P'}$ fails to have an inverse. (The prime symbol denotes transposition.)

Notes: "Block matrices" have not been treated as a topic per se, only as notation intended to save time and ink. Unlike the ordinary exam, there is no inversion here, but the matrix is bigger and with more zeroes. While the ordinary exam also had a question which called for a specific rule in showing a zero determinant by pointing out proportionality without cofactors, this set is more prescriptive in that it requires the use of Cramér. Note to that question: it has been clarified in class that asking for "an expression for", means there is absolutely no requirement to discuss the validity of the formula, so no reservation on division by zero is needed for part(c). For the last question, $\mathbf{PAA'A^3P'}$ was hopefully ugly enough to keep everyone from calculating it.

For part (a) it is of course possible to calculate $\mathbf{M}^2$ first, and that is OK (but time-consuming: it was intended to be up to them to spot that $\mathbf{M}(\mathbf{Mv})$ is quicker.)

Solution:

(a)
- $\mathbf{Mv}$ equals third row of $\mathbf{M}$, i.e. $\underline{\underline{5\mathbf{v}}}$. Then $\mathbf{M}^2\mathbf{v} = \mathbf{M}(5\mathbf{v}) = \underline{\underline{25\mathbf{v}}}$. The last matrix product $\underline{\underline{\text{does not exist}}}$ because $\mathbf{v}^2$ doesn't (orders don't match up).
- $\mathbf{I}$ to a power is $\mathbf{I}$. We get the fourth row $(0,\ 0,\ 0,\ p-5,\ q)$ of $\mathbf{M}-\mathbf{I}$ dotted with the [transpose of the] last column of $\mathbf{M}$: $(0,\ 0,\ 0,\ p-5,\ q)\cdot(0,\ 0,\ 0,\ q,\ s) = 0 + (p-5)q + qs$. Or if you like, $\underline{\underline{(p-5+s)q}}$.

(b)
- Determinant of product: $|\mathbf{PA}| = |\mathbf{A}|\ |\mathbf{P}|$. For $|\mathbf{M}|$, cofactor expansion along the third row yields $|\mathbf{M}| = 5\begin{vmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & p & q \\ 0 & 0 & r & s \end{vmatrix}$; then along first row, we get $5a\begin{vmatrix} d & 0 & 0 \\ 0 & p & q \\ 0 & r & s \end{vmatrix} - 5b\begin{vmatrix} c & 0 & 0 \\ 0 & p & q \\ 0 & r & s \end{vmatrix} = 5ad|\mathbf{P}| - 5bc|\mathbf{P}| = 5|\mathbf{A}|\ |\mathbf{P}|$ since $ad-bc = |\mathbf{A}|$.
- The third column is $5\mathbf{v}$, so replacing $\underline{\underline{\text{any other column than the third}}}$ by $\mathbf{v}$ yields two proportional columns and zero determinant.

(c)
- From the previous part, $x_1 = x_2 = x_4 = x_5 = \frac{0}{|\mathbf{A}|} = 0$. We need $x_3 = \frac{D_3}{|\mathbf{A}|}$. But $D_3 = \frac{1}{5}|\mathbf{A}|$ so $x_3 = \frac{1}{5}$. The expression is $\underline{\underline{\frac{1}{5}\mathbf{v}}}$.

  [Or, one could have started repeating the expansion along the middle row(/column!). It is OK to write out the vector, no need to write in terms of $\mathbf{v}$.]
- We see that $\frac{1}{5}\mathbf{v}$ is a solution always (from (a), $\frac{1}{5}\mathbf{Mv} = \mathbf{v}$). Thus, there are several solutions precisely when $|\mathbf{M}| = 0$, i.e. precisely when $|\mathbf{P}|\ |\mathbf{A}| = 0$. Now the matrix $\mathbf{PAA'A^3P'}$ fails to have an inverse when its determinant is zero. Its determinant is $|\mathbf{P}|\ |\mathbf{A}|\ |\mathbf{A}'|\ |\mathbf{A}|^3\ |\mathbf{P}'| = |\mathbf{P}|^2|\mathbf{A}|^5$ since $|\mathbf{A}'| = |\mathbf{A}|$ always. This is zero when precisely when $|\mathbf{P}|\ |\mathbf{A}|$ is.

Problem 3 of 5. ***Suggested weight: 15 percent***

(a)
- *Use the substitution* $u = \ln t$ to calculate $\int \ln t\, dt$.

  This particular substitution is mandatory, and you are not allowed to use any other substitution until after you are done antidifferentiating, only then are you allowed to substitute back.

  You *are* however allowed to use integration by parts after you have substituted.
- Calculate $\int_1^{e^K} (\ln t)^{1+K}\, dt$ for some constant $K > 0$ of your choice.

(b) Consider the differential equation $2\dot{x} + 3 = 4x + 6t$.
- There is a particular solution of the form $x(t) = Qt$. Find the constant Q.
- Find the general solution.

(a)
- $u = \ln t$ yields $du = \frac{dt}{t}$ and $dt = t\, du = e^u\, du$, so the integral is $\int ue^u\, du$, which we integrate by parts as $ue^u - \int 1e^u\, du = (u-1)e^u + C$. Substituting back, we get $(\ln t - 1)t + C$.

  [Like the ordinary exam, a substitution is *prescribed;* the textbook $1 \cdot \ln t$ integration by parts is simply not an answer to the question. The next bullet item can be calculated by any method though.]
- Trying the same substitution yields first the indefinite integral $\int u^{K+1}e^u\, du$. Put $K = 1$ and integrate by parts: $u^2e^u - \int 2ue^u\, du = D + u^2e^u - 2(u-1)e^u$ from the previous bullet item. Substitute back to get $D + (\ln t)^2 t - 2(\ln t - 1)t$, to be evaluated at $t = e^K = e$ and $t = 1$: $e - 2 \cdot 0 - [0 - 2 \cdot (0-1)] = \underline{\underline{e - 2}}$.

  [Notes: They are urged to do the indefinite integral first, unless they are sure how to substitute in the definite, which would lead to $\int_0^K u^{1+K}e^u\, du$ and so forth. Choosing a natural number for K to get something solvable by hand, is part of the problem. There is no extra score for showing off with a higher K than 1.]

(b)
- Insert Qt to get $2Q + 3 = 4Qt + 6t = 2(2Q+3)t$. Equality holds – as $0 = 0$ – for $\underline{\underline{Q = -3/2}}$.
- [Here one can use particular solution $-3t/2$ + general solution Ce^{2t} of the homogeneous equation $2\dot{x} = 4x$. Not relying upon that quick route:]

  Write first as $\dot{x} - 2x = 3t - 3/2$ and use either formula or integrating factor e^{-2t} to get $\frac{d}{dt}\big(xe^{-2t}\big) = (\dot{x} - 2x)e^{-2t} = (3t - \frac{3}{2})e^{-2t}$. We integrate the right-hand side by parts as vw' with $v = 3t - \frac{3}{2}$ and $w = -\frac{1}{2}e^{-2t}$:

$$xe^{-2t} = C - \tfrac{1}{2}(3t - \tfrac{3}{2})e^{-2t} + \tfrac{1}{2}\int 3e^{-2t}\, dt$$

so that $$x = Ce^{2t} - \tfrac{1}{2}(3t - \tfrac{3}{2}) + \tfrac{3}{2} \cdot \tfrac{-1}{2}e^{-2t}e^{2t} \qquad \underline{\underline{= Ce^{2t} - \tfrac{3}{2}t}}$$

Problem 4 of 5. ***Suggested weight: 25 percent*** The following Problem (P) is a model for a Pareto efficient allocation of two goods in unit supply between two agents with the same utility function:

$$\max_{x,y} h(x,y) \qquad \text{subject to} \quad \begin{cases} r(x,y) \geq 3 - C \\ 0 \leq x \leq 1, \qquad\qquad 0 \leq y \leq 1 \end{cases} \tag{P}$$

where $r(x,y) = h(1-x, 1-y)$, where $h(x,y) = x^{2\theta} + y^{2\theta} + x^{\theta\cdot(1-\theta)}y^{\theta\cdot(1+\theta)}$ as on page 1, and the constant C is $\in [0,3]$.

(a) It is a fact that the admissible set is nonempty: since $r(0,0) = h(1,1) = 3$, the point $(x,y) = (0,0)$ satisfies the constraints.
Check the other conditions of the extreme value theorem.

From now on, take for granted – whether or not the extreme value theorem applies! – that there exist optimal $x = \phi(C)$ and $y = \psi(C)$.

Take also for granted that ϕ and ψ are *continuous* functions of C.

(b) Show that there exists a $C \in (0,3)$ such that the optimal value $h(\phi(C), \psi(C)) = \sqrt{2}$.
(*Hint:* Let $V(C) = h(\phi(C), \psi(C))$. Then $V(3) = h(1,1) = 3$ because if $C = 3$ we choose $x = \phi(3) = 1$ and $y = \psi(3) = 1$, allocating everything to one agent.)

(c) True or false? «*For the Kuhn–Tucker conditions associated to problem (P) to hold true, at least two multipliers must be zero.*

(d) Does the point $(x,y) = (\frac{1}{2}, \frac{1}{2})$ satisfy the Kuhn–Tucker conditions?
(The answer may depend on C.)

Notes: Part of the problem adapted from the very last seminar set, which gave an Edgeworth box of two agents with the same utility functions (not this function though!) – and including a question (in fact, two!) on what conditions hold at the midpoint $(x, y) = (\frac{1}{2}, \frac{1}{2})$ of the box. Experience from that problem (exam autumn 2018 problem 4) is why the other agent's utility has a separate symbol here. However unlike that problem, there are two questions first, on the two "apparently non-constructive" existence results from the course. Of these, the second might be harder to spot.

Solution:

(a) h is continuous, the set is closed (weak ineq's!), and (x, y) is bounded by the square $[0, 1] \times [0, 1]$.
[Note: it is not expected in this course to give further arguments that the set is closed – though it would not hurt to point out that the functions are defined on the entire set.]

(b) $V(3) = 3 > \sqrt{2}$. If $C = 0$ then we must choose $(0, 0)$, so $V(0) = 0 < \sqrt{2}$. Because V is continuous, it does for some $C \in (0, 3)$ attain the intermediate value $\sqrt{2} \in (V(0), V(3))$. [Note: expect/require no further argument that V is continuous.]

(c) True! The constraints $x \geq 0$ and $x \leq 1$ cannot both be active, so at least one of these has a zero multiplier. And the constraints $y \geq 0$ and $y \leq 1$ cannot both be active, so at least one of these has a zero multiplier.
[Note: The reason for the "True or false?" formulation is that it is possible to formulate Kuhn–Tucker conditions for this problem with only one multiplier explicitly written – and there is a chance of that happening as the autumn 2018 solution note ($\subset$ seminar 11 solution note!) did that, top of last page. The question does not ask to formulate the Kuhn–Tucker conditions, but those who lay claim to this shortcut as justification for "False, there is only one multiplier!" should at least be able to write it in a way that distinguishes them clearly from the ignorance on how the generic formulation uses one multiplier per constraint. If you claim to have done it correctly with one multiplier you should be able to write it *correctly* with one multiplier.]

(d) At $(x, y) = (\frac{1}{2}, \frac{1}{2})$, the Lagrangian reduces to $h(x, y) + \lambda \cdot (h(1 - x, 1 - y) - 3 + C)$. The first-order conditions become $h'_x(\frac{1}{2}, \frac{1}{2}) - \lambda h'_x(\frac{1}{2}, \frac{1}{2}) = 0$ and $h'_y(\frac{1}{2}, \frac{1}{2}) - \lambda h'_y(\frac{1}{2}, \frac{1}{2}) = 0$, and because h has no stationary point, $\lambda = 1$ and active constraint. $\underline{\underline{\text{OK precisely when } C \text{ is so that the constraint is active.}}}$
[It is arguably not needed to write this out as $C = 3 - h(\frac{1}{2}, \frac{1}{2})$. And, some imprecise language on the if and only if part must be tolerated, it is hardly common to use implication arrows correctly in this course. Papers arguing like economists, yes-if-that-keeps-the-other-agent-indifferent, will likely need to be considered case by case.]

Problem 5 of 5. ***Suggested weight: 20 percent*** Let again $\theta \in (0, 1/2)$ and consider the same functions as on page 1, defined (for positive x and y) by

$$f(x,y) = x^{2\theta} + y^{2\theta}, \qquad g(x,y) = x^{\theta\cdot(1-\theta)}y^{\theta\cdot(1+\theta)}, \qquad h(x,y) = f(x,y) + g(x,y)$$

(a) What – if anything – does l'Hôpital's rule tell us about $\lim\limits_{\theta\to 0^+} \dfrac{x^{2\theta} + y^{2\theta} - 2}{x^{\theta\cdot(1-\theta)}y^{\theta\cdot(1+\theta)} - 1}$?

(b) It is possible to show that $El_x h(x,y) + El_y h(x,y)$ is a constant, *without* calculating any of these elasticities or any derivatives. How?

(c) Calculate the elasticity of substitution for f.

Notes and solutions:

(a) [To ask what l'Hôpital's rule tells us, attempts at forcing them to (I) check the validity – and someone might think it is a trap and answer "nothing, it does not apply"; and (II) actually carry it out rather than pointing at some term and say "exponential!".]
The limit is "$\frac{1+1-2}{1\cdot 1-1}$", and l'Hôpital's rule is valid. The derivative of the denominator wrt. θ is $x^{\theta\cdot(1-\theta)}y^{\theta\cdot(1+\theta)} \cdot \frac{\partial}{\partial\theta}[\theta\cdot(1-\theta)\ln x + \theta\cdot(1 \boldsymbol{+} \theta)\ln y] = x^{\theta\cdot(1-\theta)}y^{\theta\cdot(1+\theta)} \cdot [(1-2\theta)\ln x + (1+2\theta)\ln y]$. Corrected 2021: The bold-face "+".

$$\lim_{\theta\to 0^+} \frac{2x^{2\theta}\ln x + 2y^{2\theta}\ln y}{x^{\theta\cdot(1-\theta)}y^{\theta\cdot(1+\theta)}\cdot[(1-2\theta)\ln x + (1+2\theta)\ln y]} = \frac{2\ln x + 2\ln y}{1\cdot[\ln x + \ln y]} = \underline{\underline{2}}$$

(b) [Akin to problem 4 on the autumn 2014 exam, assigned for seminar 9 – but also giving the information that the sum is constant. Arguably, "By showing it is homogeneous!" would be a complete answer to the question as stated, and full score should be considered. The following verifies it – no need to point *only if* h is homogeneous.]
True if h is homogeneous. f is, so we need g homogeneous of *same* degree 2θ: $g(tx,ty) = t^{\theta(1-\theta)+\theta(1+\theta)}g(x,y)$, and $\theta(1-\theta)+\theta(1+\theta) = 2\theta - \theta^2 + \theta^2 = 2\theta$, *OK!*

(c) ln(MRS) for f is $\ln(f'_x/f'y) = \ln(2\theta x^{2\theta-1}) - \ln(2\theta y^{2\theta-1}) = (2\theta-1)[\ln x - \ln y]$, so:

$$\sigma_{xy} = \frac{d\ln(y/x)}{d\ln(x^{2\theta-1}/y^{2\theta-1})} = \frac{d\ln y - d\ln x}{(2\theta-1)d\ln x - (2\theta-1)d\ln y} = \underline{\underline{\frac{1}{1-2\theta}}}$$

Notes for (c):

- This was how the generic CES was covered in class; this simple special case might not be recognizable, but the word "Calculate" was *intended* to ask for calculating and not just looking up. Still the gradig committee should make a best judgement if someone claims to recognize it as CES with given parameter.
- Also OK: to use (even stated without source) the formula $\frac{f'_x f'_y}{xy} \cdot \frac{xf'_x + yf'_y}{\begin{vmatrix} 0 & f'_x & f'_y \\ f'_x & f''_{xx} & f''_{xy} \\ f'_y & f''_{xy} & f''_{yy} \end{vmatrix}}$ [and do the calculations].