ECON3120/4120 Mathematics 2*

December 7th 2021, 1500–1900 (4 hrs). There are 2 pages of problems to be solved.
Support material: "Rules and formulas" attachment†, and both the approved calculators.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Problem 1 Throughout Problem 1, a, b, c, d, h, u, v, w are real constants with $ad \neq bc$ and $h = \dfrac{1}{ad - bc}$. Let $\mathbf{A} = \begin{pmatrix} a & 0 & b \\ c & 0 & d \\ 0 & h & 0 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$. The prime symbol denotes transposition.

(a) For the matrix products $\mathbf{A}^2$, $\mathbf{r}'\mathbf{A}$, $\mathbf{rr}$ and $\mathbf{rr}'$: Calculate those which are well-defined, and explain why the others are not well-defined.

(b) For $\mathbf{A}$, $\mathbf{r}$ and those matrices which were well-defined in part (a):
Calculate its determinant, or explain why the determinant is not well-defined.

(c) *Use parts (a) and (b)* to decide for what value(s) of the constants (if any!) $\mathbf{A}^{-1}$ will:
(i) exist; (ii) equal $\mathbf{A}$; (iii) equal $\mathbf{A}^2$ (*hint:* determinants!)
(If you could not complete (a) and (b), you can get partial score for explaining what you would have done had you had the answers.)

(d) Suppose $\mathbf{x}$ and the constants are such that $\mathbf{x}$ is the *unique* solution of the equation system $\mathbf{Ax} = \mathbf{r}$. Find the first element x_1.

*Corrected version: This previous exams repository is intended for future students' exam preparations, and this version has been amended for that use. See the solution note.
†Same attachment as to the 2019 exam (as announced in advance). Attachment omitted from this version.

Problem 2

(a) • Show that $\int \log_B t \, dt = C + \log_B \left((t/e)^t\right)$ when the base number B equals e.

• Does the formula hold for any *other* base numbers $B > 1$?

(b) Show that there is a $T > 1$ such that $\int_{1/2}^{T} \ln t \, dt = 0$. (You are not asked to *find* T.)

(c) Does $\int_1^0 \ln t \, dt$ (i) converge or (ii) diverge to $+\infty$ or (iii) diverge to $-\infty$ or (iv) neither?

(d) Write $\left(\ln t\right)^2$ as $u(t)v'(t)$ where $u(t) = v'(t) = \ln t$, and calculate $\int \left(\ln t\right)^2 dt$ *using integration by parts.* (After integrating by parts, you can use (a).)

(e) The differential equation $\dot{x} + ax = 2^t \ln t$ can be solved (in this course) for *one* nonzero constant a. Find that a, and find the particular solution such that $x(1) = 1/a$.

Problem 3

(a) Show that $(x_1, y_1) = (1, 2)$ is a saddle point for the function $h(x, y) = xy^3 - 4 \cdot (x^2 + 3y)$.

Let $r > 0$ be a constant. Consider now the problems

$$\max \ xy^3 \quad \text{subject to} \quad \tfrac{1}{7}(x^2 + 3y) = r \qquad \text{(L)}$$

$$\max \ xy^3 \quad \text{subject to} \quad \tfrac{1}{7}(x^2 + 3y) \le r \qquad \text{(K)}$$

(b) • State the Lagrange conditions associated with problem (L).

• State the Kuhn–Tucker conditions associated with problem (K).

(c) Suppose $r = 1$. For each of the points $(x_1, y_1) = (1, 2)$ (as in (a)) and $(x_2, y_2) = (-1, 2)$:

• Verify that the point satisfies the Lagrange conditions associated with problem (L).

• Check the point against the Kuhn–Tucker conditions associated with problem (K).

(d) Points (x_1, y_1) and (x_2, y_2) both satisfy $x^2 = r$. Is there any admissible point (x, y) satisfying the Kuhn–Tucker conditions associated with problem (K), but with $x^2 \neq r$?

Consider two modifications of (K) as follows (note, point $(x, y) = (0, r)$ is admissible):

$$\max / \min \ xy^3 + \frac{(\ln r)^2}{\ln(e + x)} \quad \text{subject to} \quad \frac{x^2 + 3y}{7} \le r, \quad x \ge -\frac{e}{2}, \quad y \ge r \qquad (\mathrm{P_{max}} / \mathrm{P_{min}})$$

(e) Pick one of the problems (K), $(\mathrm{P_{max}})$, $(\mathrm{P_{min}})$ and show that it has a solution for every constant $r > 0$. (It is part of the question to select one which the course enables you to show. There could be solution to more than one.)

(f) Take for granted that when $r = 1$, point (x_1, y_1) solves problem $(\mathrm{P_{max}})$ and point (x_2, y_2) solves problem $(\mathrm{P_{min}})$. Pick one of these, and approximate how much the optimal value function changes if r changes by $1/2021$. (Do not care about the sign of the change.)

(End of problem set. Attachment: Rules and formulas.)