University of Oslo / Department of Economics

ECON3120/4120 Mathematics 2: on the 2022-01-13 exam

- *Standard disclaimer:* A note like this is not suited as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
  - The document reflects what was expected in that particular semester, and which may not be applicable to future semesters. In particular, what tests one is required to perform before answering «no conclusion» may not apply for later.
- Weighting: At the discretion of the committee (and in case of appeals: the new grading committee).

  The problem set was written with the intention that a uniform weighting over letter-enumerated items should be a *feasible* choice, and it has been communicated that so regularly happens.
- *Special considerations for 2021: new exam format per 2019.* Starting 2019 – but excluding 2020! – there is a 4 hour (changed from 3) school exam; one can no longer bring written support material (instead, there is a pre-announced known «Rules and formulas» attachment following the problem set); and, the problem set is now in English only (however there is no change in the regulations as for what languages are permitted for the submitted papers).

  As the Department decided against discussing more extensive changes, the format should not be taken to intend changes in overall requirements. Rather, there is a hope that this will facilitate better differentiation between candidates; the grading distribution may of course be affected, should the committee find it appropriate.
- *On changes in teaching format, effective 2019:* See 2019 exam guidelines.
- *Problems from compulsory hand-in sets,* cf. 2019 change in compulsory activities (again, see the 2019 guideline for general considerations): Some comments relating the exam problems to hand-in problems are given before each solution. Hand-in problem sets are available to graders upon request.

Addendum after ordinary grading (considerations in case of appeals): Apparently it was not straightforward to assign grades to this exam. Several papers were weak even when it comes to prerequisite skills like the chain rule for differentiation: although differentiating the ln seems to pose few problems, the two differentiations of a form $\frac{d}{dz}\ln(K+g(z))$ (4(a) and 5(a)) were most often answered wrong or not at all. Nevertheless, most grading thresholds were tweaked a couple of points downwards from the usual (91/75/55/45/40) in part because that move by and large distinguished stronger papers from weaker ones, in part from a benefit of doubt consideration (concerning in particular, the failing at a compulsory course – and after all, most of the papers exhibiting very weak differentiation skills still ended up E or worse).
In case of appeals, the Department is advised to assign an appeals committee of persons involved in the grading (or appeals grading) of the ordinary December 2021 exam and can compare performances fairly between the two problem sets.

Problems restated with notes and solution following (boxed).

Problem 1 Take for granted that the following equation system determines continuously differentiable functions $x = x(s,t)$ and $y = y(s,t)$ around the point where $(s,t,x,y) = (4,3,2,1)$:

$$\begin{aligned} x + y^t + \ln(x-1) + \ln y &= t \\ txy + e^{t-x-y} &= s + t \end{aligned}$$

(a) Differentiate the system (i.e., calculate differentials).

(b) Calculate $\dfrac{\partial y}{\partial t}(4,3)$.

Notes: This is routinely taught after linear algebra – to identify the resulting equation system as linear (though linear algebra language is not at all expected). A downside is that "after linear algebra" means too late to cover it in any of the hand-ins. It has been emphasized that this does not mean it is of lesser importance, and that it is often assigned on an exam and often as the first problem.
There is a "difficulty" here in differentiating y^t. The formula collection available with the exam, gives the way out: rewrite into $e^{t\ln y}$ if needed.

How to solve:

(a) Differentiating term by term:

$$\begin{aligned} dx + ty^{t-1}\,dy + y^t \ln y\; dt + \frac{1}{x-1}dx + \frac{1}{y}dy &= dt \\ xy\,dt + ty\,dx + tx\,dy + e^{t-x-y}(dt - dx - dy) &= ds + dt \end{aligned}$$

(b) We are only asked for a partial change in t, from the point. Put $ds = 0$ and insert coordinates to get

$$\begin{aligned} dx + 3\cdot 1\,dy + 0\,dt + dx + dy &= dt \\ 2\,dt + 3\,dx + 6\,dy + dt - dx - dy &= dt \end{aligned}$$

For a derivative of y, eliminate dx by subtracting equations:

$$4\,dy + 2\,dx - \big(2\,dx + 3\,dt + 5\,dy\big) = 0$$

which yields $dy = -3\,dt$ and the answer is $\underline{\underline{-3}}$.

Problem 2 For each constant t let $\mathbf{A}_t = \begin{pmatrix} 1 & 2 & 1 \\ t & -3 & 1 \\ 0 & 2 & 0 \end{pmatrix}$

(a) Calculate $\mathbf{A}_0\mathbf{A}_0'$ (this only for $t = 0$) and the determinant $|\mathbf{A}_t\mathbf{A}_t'\mathbf{A}_t|$ (The prime symbol denotes matrix transpose.)

(b) Pick a t such that $\mathbf{A}_t^{-1}$ exists (your choice!), and invert $\mathbf{A}_t$.

Notes: For (a): It is fairly new that matrix multiplication merits score in itself. It has been stressed that matrix multiplication does not commute. Knowing that the determinant equals $|\mathbf{A}_t|^3$ is expected, but one cannot penalize those who calculate the full product first. (It is a waste of exam time.)
For (b): It has been stressed that one must know more than one method to invert a matrix, and the distinction between (i) establishing existence, (ii) testing/verifying a given candidate (assigned for the ordinary exam), and (iii) computing one. For more than one seminar – including a hand-in – there have been a "your choice" of parameter, and it has been emphasized that they should choose one that makes their task easier (no credit for picking a tougher one).

How to solve:

(a) Element (i, j) of $\mathbf{A}_0\mathbf{A}_0'$ equals the dot product between row i and row j of $\mathbf{A}_0$, so it will be symmetric (the dot product between rows i and j is the same as between rows j and i). Hence the following only writes out the calculations on and above the main diagonal:

$$\begin{pmatrix} 1^2 + 2^2 + 1^2 & 0 + 2 \cdot (-3) + 1 \cdot 1 & 0 + 2 \cdot 2 + 0 \\ -5 & 0^2 + (-3)^2 + 1^2 & 0 - 3 \cdot 2 + 0 \\ 4 & -6 & 0^2 + 2^2 + 0^2 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 6 & -5 & 4 \\ -5 & 10 & -6 \\ 4 & -6 & 4 \end{pmatrix}}}$$

For the determinant of the product, it equals $|\mathbf{A}_t| \cdot |\mathbf{A}_t'| \cdot |\mathbf{A}_t| = |\mathbf{A}_t|^3$, we calculate $|\mathbf{A}_t|$. Cofactor expansion along the last row yields $|\mathbf{A}_t| = -2\begin{vmatrix} 1 & 1 \\ t & 1 \end{vmatrix} = -2(1 - t) = 2(t - 1)$. The answer is $\underline{\underline{8(t - 1)^3}}$.

(b) For Gaussian elimination, the easiest might be $t = 0$ (and, $t = 1$ is out of the question). Also it might be quicker to use the third row to eliminate second column: Subtract third row from the first, and add 3/2 of it to the second, we get $\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 3/2 \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$. Subtract the second from the first to get

$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1-{}^{3}\!/_{2} \\ 0 & 0 & 1 & 0 & 1 & {}^{3}\!/_{2} \\ 0 & 2 & 0 & 0 & 0 & 1 \end{array}\right)$. Scale the third and interchange it with row number two, to get a right-hand side of $\underline{\underline{\begin{pmatrix} 1 & -1 & -{}^{5}\!/_{2} \\ 0 & 0 & {}^{1}\!/_{2} \\ 0 & 1 & {}^{3}\!/_{2} \end{pmatrix}}}$.

Problem 3

(a) Use the substitution $u = e^x - 1$ to show that $\displaystyle\int \frac{1}{1-e^{-x}} dx = C + \ln\left|e^x - 1\right|$.
(If unable to show it by antidifferentiation using this particular substitution, show it by any means you like *for up to half score.*)

(b) Find the general solution of the differential equation $\dot{x} = (1 - e^{-x}) \cdot (-t^{-2})$.

Notes: The ordinary exam had a linear differential equation, this has a separable one. Using an antiderivative given already is quite commonplace on these exams, and it has been clarified and repeated (and stated on the exam itself) that they are free to use part (a) in part (b).
The constant solution is more than a detail in this course: not checking for constant solution means one divides by something that could be zero. (Indeed, hand-in 3 gave to show existence of a constant solution to an equation where no non-constant solution could be calculated by hand.)

How to solve:

(a) The subsitution yields $du = e^x\,dx$ and we need $1 - e^{-x} = e^{-x}(e^x - 1) = ue^{-x}$. Inserting, we have $\frac{dx}{ue^{-x}} = \frac{e^x}{u} dx = \frac{1}{u}\,du$ which has an antiderivative of $\ln|u|$. Thus the integral is $C + \ln|u| = C + \ln|e^x - 1|$, *qed.*

(b) Constant solution iff $1 = e^{-x}$ i.e. iff $x = 0$. For $x \neq 0$, separate into $\frac{1}{1-e^{-x}} dx = -t^{-2}\,dt$, integrate (using (a)) as $\ln|e^x - 1| = K + 1/t$, and solve for x. From $|e^x - 1| = e^K e^{1/t}$ we put $Q = \pm e^K$, observe that $Q = 0$ subsumes the constant solution $x = 0$, and so $e^x = 1 + Qe^{1/t}$ and thus $\underline{\underline{x = \ln\left(1 + Qe^{1/t}\right)}}$.

Problem 4 For each constant $r > 0$, define $f(t) = 2rt + \ln(2022 + \sqrt{t}) - e^{2t-\sqrt{t}}$. Here, f is defined for all $t \geq 0$.

(a) • Calculate $\lim_{t\to+\infty} f'(t)$. (Note, it is f' and not f.)

• Show that $\lim_{t\to+\infty} f(t) = -\infty$. *Hint:* $\lim_{t\to+\infty} \dfrac{f(t)}{e^t}$ is helpful, and worth partial score; consider positive and negative terms separately!

(b) Show that f has a zero z. You are *not* asked to solve $z = z_r$ out for r.

In the rest of Problem 4, take for granted that there is only one zero z for each r.

(c) Show that f has a global maximum τ, and that $\tau \in (0, z)$.

Let $A = A(r)$ be the area under the graph up to z (that is, more precisely: the area bounded by both axes, the graph of f, and the vertical line $t = z$).

(d) Calculate $A'(r)$ and express it with no integral sign. (It is possible to express $A'(r)$ in terms of z only, and again you are *not* asked to solve z out for r.)

Notes: (a)–(c) closely follow parts (a)–(c) of hand-in no. 2 for seminar 5 – as well exam autumn 2017 problem 1 (a)–(c) which was assigned for seminar 7 in order to review the very same problem type. Part (d) closely follows part (e) from the same autumn 2017 problem 1 – and like that problem, the possible contribution from the z must be noted before zeroed out.
For part (a) second bullet item, it is not sufficient to call $e^{2t-\sqrt{t}}$ nor $e^{t-\sqrt{t}}$ an "exponential" without further argument (what is that exponent?!), and the hint is intended to lead to more adequate answers. It *is* possible given the first item to get $-\infty$ by using the tangent direction, like was done in (certainly a hard part of) hand-in 2.

How to solve:

(a) • We need $f'(t) = 2r + \frac{1}{2022+\sqrt{t}} \cdot \frac{1}{2\sqrt{t}} - (2 - \frac{1}{2\sqrt{t}})e^{2t-\sqrt{t}}$, which as $t \to +\infty$ will tend to $2r - 0 - 2\lim_{t\to+\infty} e^{2t-\sqrt{t}}$. Because $2t - \sqrt{t} = (2\sqrt{t}-1)\sqrt{t}$ will $\to +\infty$ and thus the exponential will, we get $2(r - \infty) = \underline{\underline{-\infty}}$.

• $\lim_{t\to+\infty} \dfrac{f(t)}{e^t} = \lim_{t\to+\infty} \left(\dfrac{2rt + \ln(2022 + \sqrt{t})}{e^t} - e^{t-\sqrt{t}}\right)$. Consider each term: the first is «$\frac{\infty}{\infty}$», and by l'Hôpital it equals $\lim_{t\to+\infty} \dfrac{2r + \frac{1}{(2022+\sqrt{t})\cdot 2\sqrt{t}}}{e^t} = 0$. Because $t - \sqrt{t} = \sqrt{t}(\sqrt{t} - 1) \to +\infty$, then $\lim_{t\to+\infty} \left(-e^{t-\sqrt{t}}\right) = -\infty$, and $\lim_{t\to+\infty} \dfrac{f(t)}{e^t} = -\infty$. Then $\lim_{t\to+\infty} f(t) =$ is $(-\infty)\cdot(+\infty) = -\infty$.

(b) $f(0) = 0 + \ln 2022 - 1 > 0$. From (a), we know that f becomes negative somewhere. f is continuous, so by the intermediate value theorem it has a zero.

(c) By the extreme value theorem, f has a maximum τ over the closed and bounded interval $[0, z]$. Since $f(\tau) \geq f(0) > 0$, τ is a maximum also over all t, as $t > z$ yield negative f-values. It remains to show that $0 < \tau < z$, i.e. rule out the endpoint maximum. There cannot be a maxiumum at z, as $f(z) = 0$ is less than $f(0)$. From (a), we have that f increases at 0, which cannot be a maximum point.

(d) $f > 0$ on $[0, z]$, so $A(r) = \int_0^z \left(2rt + \ln(2022 + \sqrt{t}) - e^{2t-\sqrt{t}}\right) dt$. Differentiating using the Leibniz rule (noting that z depends on r!): $A'(r) = f(z)\frac{dz}{dr} + \int_0^z \left(2t + 0\right) dt = 0 + \left[t^2\right]_0^z = \underline{\underline{z^2}}$ since $f(z) = 0$ by the definition of z.

Problem 5 Let $q > 0$ be constant and let $f(x, y) = x^{5/4}y^{3/2}$. Consider the problems

$$\max\ f(x,y) \quad \text{subject to} \quad y^2 + x^8 - \ln(1 + x^3) = q \tag{L}$$

$$\max\ f(x,y) \quad \text{subject to} \quad y^2 + x^8 - \ln(1 + x^3) \leq q \quad \text{and} \quad x \leq q \tag{K}$$

Note that both x and y are nonnegative due to the exponents $5/4$ and $3/2$. You are not asked to *solve* these problems (and you should not try).

(a) State the Lagrange conditions associated with problem (L), and state the Kuhn–Tucker conditions associated with problem (K).

(b) Verify that the point $(x, y) = (0, \sqrt{q})$ satisfies the Lagrange conditions associated with (L), and check whether it satisfies the Kuhn–Tucker conditions associated with (K).

(c) The optimal value of problem (K) depends on q, call it $V(q)$. Show that $V'(q) > 0$ whenever $V'(q)$ exists. (You can calculate as if V' always exists.)

Notes: Problem 5 might be the closest one to a problem at the ordinary exam, namely the last one; that in turn closely followed parts of the first hand-in.
Notes for (c):

- Like the ordinary exam – and the hand-in – the parameter occurs twice in the Lagrangian. Final question on exam, and not easy at this level of precision. Partial score should be awarded for partial rights.

- (b) yields that the Kuhn–Tucker conditions hold at some point which has $\lambda = \mu = 0$, but that does not mean the point is optimal. Those who think first-order condition *implies* optimality, will go wrong, but many will likely (/hopefully) jump to the envelope theorem considerations.

- (Finally, a detail not at all expected – to the extent the proposed solution below deliberately omits it: The argument of "worst possible" and thus not "best possible" is false if the function is constant on the admissible set. Constraining a production function to zero output is, understandably, not much in the focus of this course. For those curious, one can for example from the point $(0, \sqrt{q})$ of part (b) move rightwards (increasing x from 0). That requires calculations to establish the direction of the inequality, so the following is more elegant: reduce y first, making both constraints hold with strict inequality, and use continuity of both functions $y^2 + x^8 - \ln(1 + x^3) - q$ and $x - q$. Then a small displacement will maintain strict inequality and thus, admissibility.)

How to solve:

(a) Let $L(x, y) = x^{5/4}y^{3/2} - \lambda \cdot \big(y^2 + x^8 - \ln(1+x^3) - q\big) - \mu \cdot (x - q)$; this function works as Lagrangian for both problems if we set $\mu = 0$ for the (L) problem. We need partial derivatives $L'_x(x, y) = \frac{5}{4}x^{1/4}y^{3/2} - \lambda \cdot \big(8x^7 - \frac{3x^2}{1+x^3}\big) - \mu$ and $L'_y(x, y) = \frac{3}{2}x^{5/4}y^{1/2} - 2\lambda y$.

The Lagrange conditions (putting $\mu = 0$):

$$\tfrac{5}{4}x^{1/4}y^{3/2} - \lambda \cdot \big(8x^7 - \tfrac{3x^2}{1+x^3}\big) = 0 \tag{1}$$

$$\tfrac{3}{2}x^{5/4}y^{1/2} - 2\lambda y = 0 \tag{2}$$

$$y^2 + x^8 - \ln(1 + x^3) = q \tag{3}$$

The Kuhn–Tucker conditions:

$$\tfrac{5}{4}x^{1/4}y^{3/2} - \lambda \cdot \big(8x^7 - \tfrac{3x^2}{1+x^3}\big) = \mu \tag{4}$$

$$\tfrac{3}{2}x^{5/4}y^{1/2} - 2\lambda y = 0 \tag{5}$$

$$\lambda \geq 0 \quad \text{and with } \lambda = 0 \text{ if } \quad y^2 + x^8 - \ln(1 + x^3) < q \tag{6}$$

$$\mu \geq 0 \quad \text{and with } \lambda = 0 \text{ if } \quad x < q \tag{7}$$

(b) Inserting point $(x, y) = (0, \sqrt{q})$ into the Lagrange conditions, (1) simplifies to $0 = 0$, (2) to $\lambda = 0$, and for (3), the left-hand side is $(\sqrt{q})^2 + 0 - \ln 1 = q$ as it should, so the Lagrange conditions hold with $\lambda = 0$.

Then the Kuhn–Tucker conditions also hold with $\mu = \lambda = 0$.

(c) q occurs in both constraints: the envelope theorem yields $V'(q) = \lambda + \mu$. We must show that the multipliers cannot both be zero *in optimum,* so suppose for contradiction that they are. But then $x = 0$ or $y = 0$, which yields the *worst* possible output $f(x, y) = 0$, not the best.