University of Oslo / Department of Economics

ECON3120/4120 Mathematics 2: on the 2021-12-07 exam

This is a reworked version for use in future exam preparations;
After the exam, but before the grading committee had commenced its work, it was discovered an error in the exam: interchanged symbols, apparently affecting part of 1(c). The actual guideline therefore had to be updated with a lot of details on potential consequences and suggestions on how to check if they materialized and if so, how to handle them – including, complete enough for a grading appeals committee who would only see a small selected subset of the papers.
Ultimately the issue was not hard to rectify in grading.

As this repository of old exams serves the purpose of supporting preparations for future exams, and the considerations concerning error handling would significantly reduce the signal-to-noise ratio of the document, this version presents what should have been (and so also for the problem set).

As this is not the actual grading guideline document, the preamble is also removed. Several of the considerations for the 2019 exam (which was the first in this 4-hour format) would still apply. Also some notes after each problem are shortened down.

Problems restated and solutions and annotations (boxed) follow:

Problem 1

Throughout Problem 1, a, b, c, d, h, u, v, w are real constants with $ad \neq bc$ and $h = \dfrac{1}{ad-bc}$. Let $\mathbf{A} = \begin{pmatrix} a & 0 & b \\ c & 0 & d \\ 0 & h & 0 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$. The prime symbol denotes transposition.

(a) For the matrix products $\mathbf{A}^2$, $\mathbf{r}'\mathbf{A}$, $\mathbf{rr}$ and $\mathbf{rr}'$: Calculate those which are well-defined, and explain why the others are not well-defined.

(b) For $\mathbf{A}$, $\mathbf{r}$ and those matrices which were well-defined in part (a):
Calculate its determinant, or explain why the determinant is not well-defined.

(c) *Use parts (a) and (b)* to decide for what value(s) of the constants (if any!) $\mathbf{A}^{-1}$ will:
(i) exist; (ii) equal $\mathbf{A}$; (iii) equal $\mathbf{A}^2$ (*hint:* determinants!)
(If you could not complete (a) and (b), you can get partial score for explaining what you would have done had you had the answers.)

(d) Suppose $\mathbf{x}$ and the constants are such that $\mathbf{x}$ is the *unique* solution of the equation system $\mathbf{Ax} = \mathbf{r}$. Find the first element x_1.

Problem 1 solved:

(a) $$\mathbf{A}^2 = \begin{pmatrix} a & 0 & b \\ c & 0 & d \\ 0 & h & 0 \end{pmatrix}\begin{pmatrix} a & 0 & b \\ c & 0 & d \\ 0 & h & 0 \end{pmatrix} = \begin{pmatrix} a^2+0+0 & 0+0+hb & ab+0+0 \\ ca+0+0 & 0+0+hd & cb+0+0 \\ 0+hc+0 & 0+0+0 & 0+hd+0 \end{pmatrix} = \underline{\underline{\begin{pmatrix} a^2 & hb & ab \\ ac & hd & bc \\ hc & 0 & hd \end{pmatrix}}}$$

$$\mathbf{r}'\mathbf{A} = (u \quad v \quad w)\begin{pmatrix} a & 0 & b \\ c & 0 & d \\ 0 & h & 0 \end{pmatrix} = (ua+vc+0 \quad 0+wh \quad ub+vd+0) = \underline{\underline{(ua+vc \quad wh \quad ub+vd)}}$$

$\mathbf{rr}$ does not exist as $\mathbf{r}$ is not square: $m \times 1$ by $m \times 1$ does not exist for $m \neq 1$.
$\mathbf{rr}'$ exists and is 3×3, with element $(i, j) =$ the i^{th} element of $\mathbf{r}$, times the j^{th} element; that is, $\mathbf{rr}' = \begin{pmatrix} uu & uv & uw \\ vu & vv & vw \\ wu & wv & ww \end{pmatrix} = \underline{\underline{\begin{pmatrix} u^2 & uv & uw \\ uv & v^2 & vw \\ uw & vw & w^2 \end{pmatrix}}}$

(b) Only the square matrices have determinants, and those are the following: $\mathbf{rr}'$, which has proportional columns (each a scaling of $\mathbf{r}$) and thus determinant $\underline{\underline{= 0}}$. Then $|\mathbf{A}| = -h\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -h \cdot (1/h) = \underline{\underline{-1}}$ and $|\mathbf{A}^2| = |\mathbf{A}|^2 = \underline{\underline{1}}$.

(c) $|\mathbf{A}| = -1 \neq 0$ so $\underline{\underline{\mathbf{A}^{-1} \text{ exists}}}$, for all values of the constants.
The inverse $\underline{\text{cannot equal } \mathbf{A}^2}$: $|\mathbf{A}^2| = |\mathbf{A}|^2$ is always ≥ 0, while $|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$ is $= -1$ for this matrix.
$\mathbf{A}^{-1} = \mathbf{A}$ iff $\mathbf{A}^2 = \mathbf{I}$, and then (element (1,3)) $c = 0$ and (element (1,2)) $b = 0$; if $b = c = 0$, we have all zeroes outside the main diagonal, and we also have $h = 1/ad$ and thus $hd = 1/a$. The main diagonal elements are a^2, $1/a$ and $1/a$, all of which being $= 1$ iff $a = 1$. To conclude, $\mathbf{A}^{-1} = \mathbf{A}$ $\underline{\text{if and only if } a = 1 \text{ and } b = c = 0.}$

(d) Cramér's rule: $x_1 = \frac{1}{|\mathbf{A}|} \cdot \begin{vmatrix} u & 0 & b \\ v & 0 & d \\ w & h & 0 \end{vmatrix} = \frac{-h\begin{vmatrix} u & b \\ v & d \end{vmatrix}}{-1} = \underline{\underline{(ud - bv)h}}$ (or, written out as $\frac{ud-bv}{ad-bc}$).
Note, for those who want to solve without Cramér: The equation system determines $x_2 = w/h$, and there remains an equation system $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$. If $b \neq 0$, we subtract d/b of the first from the second, yields $(ad/b - c)x_1 = ud/b - v$ i.e. $(ad - bc)x_1 = ud - bv$ and the answer follows. It remains to check case $b = 0$. Since $ad \neq bc$, then d must be $\neq 0$ in the case $b = 0$. Subtract instead b/d of the second equation from the first, to get $(a - bc/d)x_1 = u - vc/d$. Multiply by hd and the answer follows.

Problem 1 notes: *First: «Any reasonable» notation for vectors or matrices is allowed. Even overarrow notation for matrices (capitals), e.g.* $\vec{C}\vec{x} = \vec{d}$.

Matrices with parameters, and equation systems with parameters, have been common in the course since forever. The notion that calculating matrix products merits score by itself is more recent. It used not to be the case. Matrix multiplication was twenty percent of hand-in number 4. The rest of linear algebra was taught too late to be covered in compulsory hand-ins, but was covered in seminar problems.

Part (a) has a potential trap in inviting those who mistake element-wise multiplications for matrix products, to actually get an answer.

Part (b) does – like the first exam in the new format – have a slight twist compared to most similar problems: it asks to point out that only square matrices have well-defined determinant. There are several rules that will lead to $|\mathbf{rr}'| = 0$*, the above only uses one of them.*

Part (c) has three questions, likely in order of increasing difficulty. Also, prescription of allowed information is often ignored – sometimes because problems are not read well enough (hence the text in emphasis), sometimes for a fond hope that it might be close enough.

In part (d), those who prefer not using Cramér face an additional obstacle in having to ensure they don't divide by zero. Division by zero isn't just a minor error in this course.

Problem 2

(a) • Show that $\int \log_B t\,dt = C + \log_B\big((t/e)^t\big)$ when the base number B equals e.

• Does the formula hold for any *other* base numbers $B > 1$?

(b) Show that there is a $T > 1$ such that $\int_{1/2}^{T} \ln t\,dt = 0$. (You are not asked to *find* T.)

(c) Does $\int_1^0 \ln t\,dt$ (i) converge or (ii) diverge to $+\infty$ or (iii) diverge to $-\infty$ or (iv) neither?

(d) Write $\big(\ln t\big)^2$ as $u(t)v'(t)$ where $u(t) = v'(t) = \ln t$, and calculate $\int \big(\ln t\big)^2 dt$ *using integration by parts.* (After integrating by parts, you can use (a).)

(e) The differential equation $\dot{x} + ax = 2^t \ln t$ can be solved (in this course) for *one* nonzero constant a. Find that a, and find the particular solution such that $x(1) = 1/a$.

Problem 2 notes *It is deliberate to ask for manipulations using basic log rules – hand-in 3 part (b) asked to show* $\int_0^T t\ln(bt^a)dt = T^p(q\ln T + r)$ *which would lead to both differentiation of the given right-hand side, log manipulation to find constants* p, q, r*, and the limit transition to zero – arguably harder than (a) and (c) combined. However, differentiation with the chain rule can lead to* $\frac{1}{(t^t e^{-t})\ln B}\frac{d(t^t e^{-t})}{dt}$ *and then diferentiation ... even if using log-differentiation on the latter is an unnecessary back-and-forth, it can hardly be penalized.*
The change of base is arguably not the most stressed part of the course, but it was essential to handle hand-in 1 part (g) – there the function even had x *in the base. Also it can be found in the formula collection along with how to differentiate it (following (C3)).*
As for (b), it is not common in the exams to have a question on the intermediate value theorem inside an integration/diff.eq. problem – but hand-in 2 part (b) had precisely that.

Problem 2 How to solve:

(a) We have $\log_B\big((t/e)^t\big) = t\log_B(t/e) = t\big(\log_B t - \log_B e\big) = \frac{1}{\ln B}\cdot t\cdot\big(\ln t - 1\big)$, and differentiating it: $\frac{1}{\ln B}\cdot\big(\ln t - 1 + t/t\big) = \frac{\ln t}{\ln B} = \log_B t$. True for all $B > 1$.

(b) The integral is a continuous function of T, and we want to use the intermediate value theorem. As $\ln t < 0$ iff $t < 1$, the integral is negative for $T = 1$ and increases as T increases from 1. Because $\ln((t/e)^t)$ (antiderivative taken from (a)) will $\to +\infty$ as $t \to +\infty$, the integral becomes positive for large enough T, and will have to hit zero for some $T > 1$.
Notes: It is not enough to say merely that the integrand/integral increases when $T > 1$ (what if it would converge?). But one can say that for $t \geq e$, $\ln t \geq 1$ and draw a sketch that indicates a contribution from an area $> 1\times$[arbitrarily big].

(c) We need $\lim_{t\to 0^+} \ln((t/e)^t) = \lim_{t\to 0^+}\big(t(\ln t - 1)\big) = \lim_{t\to 0^+}\frac{\ln t}{t^{-1}} - 0$. An «$\infty/\infty$», l'Hôpital's rule yields $\lim_{t\to 0^+}\frac{t^{-1}}{-t^{-2}} = \lim_{t\to 0^+}(-t) = 0$. So the limit exists, and the integral converges.
Notes: The limit can be inferred from (B5) in the formula collection. It was deliberate to write the integral from 1 to 0 to trap those who think such a thing cannot exist.

(d) From (a), we can use $t(\ln t - 1)$ for v. We have $\int uv'\,dt = uv - \int u'v\,dt = \ln t\cdot t(\ln t - 1) - \int t^{-1}t(\ln t - 1)\,dt = t\cdot(\ln t - 1)\ln t + t - \int \ln t\,dt = t\cdot\big[(\ln t - 1)\ln t + 1 - (\ln t - 1)\big] + C$ (from (a) again). Note, the bracketed term is $(\ln t)^2 - 2\ln t + 2 = 1 + (\ln t - 1)^2$ if so one prefers.

(e) We have (note, formula is of course OK) $\frac{d}{dt}\big(xe^{at}\big) = (\dot{x} + ax)e^{at} = e^{at}2^t\ln t$, and it all simplifies if $e^{at}2^t \equiv 1$ i.e. $at + t\ln 2 \equiv 0$ i.e. $a = -\ln 2$ so $e^{at} = (e^{\ln 2})^{-t} = 2^{-t}$. Then $x\cdot 2^{-t} = \int \ln t\,dt = C + t\big(\ln t - 1\big)$ (as above), with for the particular solution $C = x(1)2^{-1} - 1\cdot(\ln 1 - 1) = \frac{1}{2a} + 1 = 1 - \frac{1}{2\ln 2}$. This gives $x(t) = 2^t\cdot\big(1 - \frac{1}{2\ln 2} + t(\ln t - 1)\big)$.

Problem 3

(a) Show that $(x_1, y_1) = (1, 2)$ is a saddle point for the function $h(x, y) = xy^3 - 4 \cdot (x^2 + 3y)$.

Let $r > 0$ be a constant. Consider now the problems

$$\max\ xy^3 \quad \text{subject to} \quad \tfrac{1}{7}(x^2 + 3y) = r \tag{L}$$

$$\max\ xy^3 \quad \text{subject to} \quad \tfrac{1}{7}(x^2 + 3y) \le r \tag{K}$$

(b)
- State the Lagrange conditions associated with problem (L).
- State the Kuhn–Tucker conditions associated with problem (K).

(c) Suppose $r = 1$. For each of the points $(x_1, y_1) = (1, 2)$ (as in (a)) and $(x_2, y_2) = (-1, 2)$:
- Verify that the point satisfies the Lagrange conditions associated with problem (L).
- Check the point against the Kuhn–Tucker conditions associated with problem (K).

(d) Points (x_1, y_1) and (x_2, y_2) both satisfy $x^2 = r$. Is there any admissible point (x, y) satisfying the Kuhn–Tucker conditions associated with problem (K), but with $x^2 \neq r$?

Consider two modifications of (K) as follows (note, point $(x, y) = (0, r)$ is admissible):

$$\max / \min\ xy^3 + \frac{(\ln r)^2}{\ln(e + x)} \quad \text{subject to} \quad \frac{x^2 + 3y}{7} \le r, \quad x \ge -\frac{e}{2}, \quad y \ge r \qquad (\mathrm{P}_{\max} / \mathrm{P}_{\min})$$

(e) Pick one of the problems (K), $(\mathrm{P}_{\max})$, $(\mathrm{P}_{\min})$ and show that it has a solution for every constant $r > 0$. (It is part of the question to select one which the course enables you to show. There could be solution to more than one.)

(f) Take for granted that when $r = 1$, point (x_1, y_1) solves problem $(\mathrm{P}_{\max})$ and point (x_2, y_2) solves problem $(\mathrm{P}_{\min})$. Pick one of these, and approximate how much the optimal value function changes if r changes by 1/2021. (Do not care about the sign of the change.)

Problem 3 notes and solution: Problem 3 is very close to a subset of hand-in 1, but with y rescaled; the hand-in started with the function $xy^3 - 108(x^2 + y)$, the points to test were $(\pm 1, 6)$, and the constraints were $x^2 + y = 7$ resp. ≤ 7. The written solution produced for the students afterwards, had a number of colour-highlighted «Did you remember» points to check off, and some of them are tested in this exam problem, indicated with bullet items in the notes given to each part:

(a) We need the derivatives $h'_x(x, y) = y^3 - 8x$ and $h'_y(x, y) = 3xy^2 - 12$. We need the point to be stationary: $h'_x(1, 2) = 8 - 8 = 0$ and $h'_y(1, 2) = 3 \cdot 1 \cdot 4 - 12 = 0$, *OK.* We need the second derivatives $h''_{xx} = -8x$ (negative at x_1) and $h''_{yy} = 6xy$ (positive at (x_1, y_1)). The «$AC - B^2$» will therefore be negative no matter what the cross derivative is. We have a saddle point.

Note: The hand-in asked for other stationary points (there are none), but only asking for $(1, 2)$ is not only a simplification but also an intentional trap: the hand-in solution had the following bullet item:

- «Did you remember: For a point to be a saddle point, then it has to be a stationary point? [...]»
  ... and also one on cross derivatives:
- «Did you remember: That the cross derivative *is* part of the theory, and that the only reason I did not calculate it in the above, is that we already knew that AC<0 would lead to negative Hessian no matter what B?»

(b) With Lagrangian $L(x, y) = xy^3 - \lambda(\frac{x^2+3y}{7} - r)$, $L'_x(x, y) = y^3 - \frac{2\lambda}{7}x$, $L'_y(x, y) = 3xy^2 - \frac{3\lambda}{7}$, the Lagrange conditions are (0)–(2) while the Kuhn–Tucker conditions are (1)–(3):

$$\frac{1}{7}(x^2 + 3y) = r \tag{0}$$

$$y^3 - \frac{2\lambda}{7}x = 0 \tag{1}$$

$$3xy^2 - \frac{3\lambda}{7} = 0 \tag{2}$$

$$\lambda \geq 0, \text{ and if } \frac{1}{7}(x^2 + 3y) < r: \quad \lambda = 0. \tag{3}$$

Note: The hand-in part (b) also asked the Lagrange / Kuhn–Tucker conditions.

(c) (x_1, y_1): Putting $r = 1$, condition (0) holds as $1^2 + 3 \cdot 2 = 7$. From (a), the Lagrangian has a stationary point if $\lambda/7 = 4$ i.e. $\lambda = 28$, confirming the Lagrange conditions. Since $\lambda \geq 0$ and the constraint is active, the Kuhn–Tucker conditions hold at (x_1, y_1).
(x_2, y_2): Put $r = 1$ and insert for the point. Condition (0) holds as $(-1)^2 + 3 \cdot 2 = 7$. $L'_x(-1, 2) = 8 - \frac{2\lambda}{7}(-1)$ i.e. $\lambda = -28$. For (2) to hold, λ must be $7x_2y_2^2 = -28$, so the Lagrange conditions hold. Since $\lambda < 0$, the Kuhn–Tucker conditions fail at (x_2, y_2).

Note: The hand-in part (c) asked the same (up to the rescaling of y), and with the same conclusions. The solution to hand-in part (d) (not (c)!) had three bullet item mementos, the following two relevant here:

- «Did you remember to check λ?
- Did you remember that it is indeed OK to have $\lambda = 0$ for an active constraint?»

(d) Stationary points for the objective function will satisfy the Kuhn–Tucker condtions with $\lambda = 0$ (provided admissible for the constraints), and because $x = y = \lambda = 0$ hold in (1)–(3), the answer is yes, there is some other point.

Note: The hand-in part (e) asked for all *points that satisfy Kuhn–Tucker without satisfying Lagrange, a harder question that would require to find all admissible points with $y = 0$. Still it is possible to err on this by dividing by zero. The solution had a warning for (d) and (e):*

- «Did you remember that you cannot divide by zero? Surely you did, but did you remember that you cannot divide by something that *could be zero even if it isn't written by means of a circular character?*
  That is: that you cannot divide by y^2 without checking whether it could be 0? [...]»

(e) The constraints are satisfied for points that satisfy all the following

- on or below the concave parabola $y = (7r - x^2)/3$, with max point at $(0, 7r/3)$
- on or above the horizontal line $y = r$, so by now we know that the set is bounded.
- on or to the right of the vertical line $x = -e/2$.

So the set is bounded. It is closed, as boundary points are admissible (the functions in the constraints are continuous and defined everywhere). To apply the extreme value theorem, we need the objective function continuous on the set, which holds iff $\ln(e + x)$ is defined and nonzero; that holds because $x \geq -(e/2)$ so $e + x \geq e/2 > 1$. So ($\mathrm{P}_{\max}$) has a solution and so does ($\mathrm{P}_{\min}$).

Note: The hand-in had two relevant parts: Hand-in part (f) asked about existence of solution to problems (K) and (L) (there aren't, let $x \to -\infty$ along the parabola); Then hand-in part (g) had only a max problem and asked to show that it had a solution (conclusion given) – but the function was slightly different: the «new» term was a log with base $e + x$; here it is written out, as problem 2 above already tests log manipulation. The hand-in had $x \geq 0$,

$y \geq 0$ whereas this problem has $y \geq r$ and $x \geq -e/2$ (requiring the fact that $e/2 > 1$); in return, the exam problem set gives the information that the admissible set is nonempty.
The solution to the hand-in part (g) gave the following warnings, all relevant to this exam question (though the first half of the fourth bullet item – rewriting the log – is not relevant here, only to this exam's 2(a)):

- «Did you remember to read the problem well enough to understand that you were not asked for where the solution is?
- Did you remember the extreme value theorem, at all?
- Did you remember that you have to check for well-definedness of the function?
- Did you remember rewrite the log in terms of ln (or other "known base number") - and then check that you did not get the dreaded division by zero? »

(f) Consider the Lagrangian $xy^3 + \frac{(\ln r)^2}{\ln(e+x)} - \lambda(\frac{x^2+3y}{7} - r) + \alpha x + \beta(y - r)$; at the point(s), $y = 2 > 1 = r$ and $x > -e/2$, so $\alpha = \beta = 0$. Differentiate the Lagrangian wrt. r to get $\frac{\ln r}{\ln(e+x)} \cdot \frac{2}{r} + \lambda$; at $r = 1$, this reduces to λ, which is 28. So the answer is $\underline{\underline{28/2021}}$.

Notes: The last question on the hand-in was similar, albeit with two parameters (and a bug, that hardly affected anyone's workflow). Here there is one r that appears twice – also in the objective function, making it a harder question than if it would only appear as a right-hand side of a constraint.
A complete answer needs to write down the $\frac{\ln r}{\ln(e+x)} \cdot \frac{2}{r}$ term before zeroing it out; the question was deliberately constructed so that this term should vanish, so that neither those who remember nor those who forget that term, should have to insert for point coordinates. While omitting the term is an error in itself, it is an error that hardly makes for any undue simplification of the remaining part, which is to identify the numerical value of λ and scaling by the increment.
Finally: The reason why the problem says not to care about the sign of the change, is for those who – for whatever reason! – should choose (x_2, y_2). Whether or not they choose to rewrite the minimization problem into the standard form maximization problem, they should not need to worry over what sign was intended, and especially not at the very end of the exam set.