# ECON3120/4120: autumn 2022 postponed exam, solved

No formal guidelines were produced prior to grading. In case of appeals, an appeals committee should consult the Department; note however, that problem 4(c) was not intended to be this much work, and this was known to the original committee which could then exercise appropriate judgement (expect the right method but not the right calculations) – and so should any appeals committee do.
This document gives a suggested solution and some further comments.

## Problem 1:

(a) [whatever]×1 by 3×[whatever] is not a defined product, so $5\mathbf{vA}$ is undefined.

The two others are well-defined. For the first of these, writing out the terms in detail:

$$4\mathbf{Av} = 4\begin{pmatrix} 1 & 1 & 3 \\ 2 & 3 & 6 \\ 2h & -1 & 5h \end{pmatrix}\begin{pmatrix} 3 \\ 0 \\ -1 \end{pmatrix} = 4\begin{pmatrix} 1\cdot 3+0+3\cdot(-1) \\ 2\cdot 3+0+3\cdot(-1) \\ 2h\cdot 3+0+5h\cdot(-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4h \end{pmatrix}$$

$$\begin{aligned} 3\mathbf{AA}' &= 3\begin{pmatrix} 1 & 1 & 3 \\ 2 & 3 & 6 \\ 2h & -1 & 5h \end{pmatrix}\begin{pmatrix} 1 & 2 & 2h \\ 1 & 3 & -1 \\ 3 & 6 & 5h \end{pmatrix} \\ &= 3\begin{pmatrix} 1^2+1^2+3^2 & 23 & 17h-1 \\ 2\cdot 1+3\cdot 1+6\cdot 3 & 2^2+3^2+6^2 & 34h-3 \\ 2h\cdot 1-1\cdot 3+3h\cdot 6 & 2h\cdot 2-1\cdot 3+5h\cdot 6 & (2h)^2+(-1)^2+(5h)^2 \end{pmatrix} \\ &= \begin{pmatrix} 33 & 69 & 51h-3 \\ 69 & 147 & 102h-9 \\ 51h-3 & 102h-9 & 87h^2+3 \end{pmatrix} \end{aligned}$$

A note here on what to include: it would be good to at least include how one element or row or column appears, and one element that isn't "so obviously zero" that the grader cannot tell whether you multiplied the right way. In this problem set, at least $\mathbf{AA}'$ has so many elements that a grader might very well accept a right answer written straight out then (a "how could you possibly get everything right if you cannot do it right?"). But on the other hand, if you mess up calculations you still want to convince a grader that you know the method.

Arguably, the calculations above are then much more than required. Yet a remark on why the second to last matrix has the elements above the main diagonal written straight in: $\mathbf{AA}'$ is always a symmetric matrix, so if you calculate the first column first and then get to element $(1, 2)$, you will know it equals the $2\cdot 1+3\cdot 1+6\cdot 3$ from element $(2, 1)$, and can just simplify it to 23 rather than calculate it anew.

(b) [Note about calculating the determinant: It is fine to calculate it under (b), even though the solution to follow will use a more direct approach that does not utilize the determinant – not until part (c) where it turns out you will use it anyway. If you prefer to calculate it first, say, cofactor expand along the first row will get you $\left|\begin{smallmatrix} 3 & 6 \\ -1 & 5h \end{smallmatrix}\right| - \left|\begin{smallmatrix} 2 & 6 \\ 2h & 5h \end{smallmatrix}\right| + 3\left|\begin{smallmatrix} 2 & 3 \\ 2h & -1 \end{smallmatrix}\right| = 15h + 6 - (-2h) + 3(-6h - 2) = -h$. So an inverse exists for all nonzero $h$, which in part (b) leaves us with only $h = 0$ to check.]

When $h = 0$ we see from from (a) that $4\mathbf{Av} = \mathbf{0}$ and so $\mathbf{v}$ and $4\mathbf{v}$ are solutions. Note, $\mathbf{v} \neq \mathbf{0}$ so these are two distinct solutions – and more than one, that is "several".

So, put $h = h^* = 0$ and the equation system becomes

$$\begin{pmatrix} 1 & 1 & 3 \\ 2 & 3 & 6 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_2 = 0$ by the last line. Inserting $x_2 = 0$, the two first equations reduce to $x_1 = -3x_3$ and $2x_2 = -6x_3$, the latter being a scaling of the former. So we have a solution with $x_3$ chosen freely, and then the rest will be given: there is *one* degree of freedom.

(c) The second alternative is correct. If one already calculated the determinant $|\mathbf{A}_h| = -h$, the likely most straightforward argument is: $\mathbf{M}^{-1}$ exists if and only if $0 \neq |\mathbf{M}|$, and $|\mathbf{M}| = |h\mathbf{A}_h| = h^3 \cdot (-h)$, which is nonzero for all but one value (namely 0) of $h$.

Note, $h^3|\mathbf{A}_h|$ with power 3 as the matrices are $3 \times 3$. It is an error to miss the "3" although it does not alter the conclusion for this problem.

One can also take note that even without having done part (a) or (b), we can rule out the first alternative: for $h = 0$, $\mathbf{M} = h\mathbf{A}_h = \mathbf{0}$ which does not have any inverse. (It is *also* so that $\mathbf{A}_0$ fails to have an inverse, but we can rule out the first alternative even without that fact.) Knowing that there is no inverse for $h = 0$, the case $h \neq 0$ needs to be checked: the $\mathbf{M}^{-1}$ will exist (and be equal to $\frac{1}{h}\mathbf{A}_h^{-1}$) if and only if $\mathbf{A}_h^{-1}$ exists. At this stage we likely want to calculate the determinant of $\mathbf{A}_h$ even if we did not do so in part (b). It is of course fine to use any valid method to calculate the determinant, including the calculation given in the note to part (b), but the sharp eye could want to expand along the third row to *isolate the h terms* – beware the sign of the second term even though it happens to cancel: $2h\left|\begin{smallmatrix} 1 & 3 \\ 3 & 6 \end{smallmatrix}\right| - (-1)\left|\begin{smallmatrix} 1 & 3 \\ 2 & 6 \end{smallmatrix}\right| + 5h\left|\begin{smallmatrix} 1 & 1 \\ 2 & 3 \end{smallmatrix}\right|$. The second of these determinants vanishes, and if one has already spotted that $h = 0$ should yield zero, it should; if it didn't, we must have done something wrong. The terms sum to $h\big[2(6 - 9) + 5(3 - 2) = h \cdot (-1)$ which is nonzero for all nonzero $h$, in which case $\mathbf{M}^{-1}$ exists. The answer to part (c) follows.

(d) The problem has free choice among all nonzero $h$, and we would not be surprised if the vast majority chose an integer, 1 or possibly $-1$. Both formula and Gaussian elimination are perfectly valid. The following solves $\mathbf{A}_1\mathbf{X} = \mathbf{I}$ by Gaussian elimination.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 2 & 3 & 6 & 0 & 1 & 0 \\ -2 & -1 & -5 & 0 & 0 & 1 \end{array}\right) \text{ $(-2)$ of first line to second, $(+2)$ of first to third}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 1 & 2 & 0 & 1 \end{array}\right) \text{ Subtract second from last}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 4 & -1 & 1 \end{array}\right) \text{ Subtract from 1st line: second line + 3 of the last line}$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & 2 & -3 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 4 & -1 & 1 \end{array}\right) \text{ left-hand side $= \mathbf{I}$, so right-hand side = the solution.}$$

## Problem 2

(a)
- The constraints, $a \leq x_i \leq b$ for all $i = 1, 2, 3$ and $x_1 + x_2 + x_3 \leq M$, define a closed and bounded set (and nonempty, which is at worst a minor issue in this course). The objective function is continuous on this set; the ln function will become undefined "at $\ln 0$", but because $x_i \geq a > 0$, $\ln(1 + x_i)$ only takes in values $> 1$, and so each term is continuous. Hence, by the extreme value theorem, there does exist a maximium (and also a minimum).
- Moreover, the objective function is increasing in both $x_1$, $x_2$ and $x_3$, so *if we can* we will increase at least one of them. We cannot choose all of them $= b$, for then the sum would be $3b$ and we are given that $3b > M$. So if $x_1 + x_2 + x_3 < M$, then at least one of the $x_i$ would have to be $< b$ and we can increase capacity one month (or more) and profit strictly more. Hence a point with $x_1 + x_2 + x_3 < M$ can not be optimal.

(b) The problem is "almost symmetric" only with different weights on the profit-function for each month. To simplify (in this note, it is not expected on the exam) we introduce $\alpha_1 = 4$, $\alpha_2 = 4$ and $\alpha_3 = 2$ to write the Lagrangian as

$$\mathcal{L} = \sum_{i=1}^{3} \alpha_i \ln(x_i + 1) - \lambda \left( \sum_{i=1}^{3} x_i - M \right) + \sum_{i=1}^{3} \gamma_i (x_i - a) - \sum_{i=1}^{3} \mu_i (x_i - b)$$

- Conditions. Three first-order conditions, one for each variable:

$$\frac{\alpha_i}{x_i + 1} - \lambda + \gamma_i - \mu_i = 0 \quad \text{for } i = 1, 2, 3$$

and seven complementary slackness conditions, one for each multiplier:

$$
\begin{aligned}
&\lambda \geq 0 \text{ with } \lambda = 0 \text{ if } x_1 + x_2 + x_3 < M \\
&\gamma_i \geq 0 \text{ with } \gamma_i = 0 \text{ if } x_i > a \quad (\text{for each } i = 1, 2, 3) \\
&\mu_i \geq 0 \text{ with } \mu_i = 0 \text{ if } x_i < b \quad (\text{for each } i = 1, 2, 3)
\end{aligned}
$$

- Because the constraints $x_i \geq a$ and $x_i \leq b$ cannot *both hold with equality,* one of the associated multipliers (or potentially both) must be zero:

  If $x_i > a$ then $\mu_i = 0$ (and the product $\gamma_i \cdot \mu_i$ will be 0). For $\mu_i$ to be nonzero, we must have $x_i = a$. But then $x_i < b$ and thus $\gamma_i = 0$ and again the product is zero.

(c) We *try* to find a point on this form, with the monthly constraints $a \leq x_i \leq b$ all inactive (so the corresponding multipliers are zero). Furthermore, we try one that has a chance to *actually be optimal,* so that (from (a) $x_1 + x_2 + x_3 = M = 97$) – however you can see already from the following three first-order conditions that $\lambda$ cannot be zero, and thus $x_1 + x_2 + x_3$ must be 97 indeed:

$$\frac{4}{x_1 + 1} = \lambda, \qquad \frac{4}{x_2 + 1} = \lambda, \qquad \frac{2}{x_3 + 1} = \lambda$$

An elegant way to proceed is to rewrite them as $4 = \lambda(x_1 + 1)$, $4 = \lambda(x_2 + 1)$ and $2 = \lambda(x_3 + 1)$, and add these three up into $10 = \lambda \cdot (x_1 + x_2 + x_3 + 3)$. Inserting $x_1 + x_2 + x_3 = 97$, this says $100\lambda = 10$ and so $\lambda = \frac{1}{10}$ (nonnegative, *OK!*) and the first-order conditions yield $4 = \frac{1}{10}(x_1 + 1)$, $4 = \frac{1}{10}(x_2 + 1)$, $2 = \frac{1}{10}(x_3 + 1)$, i.e. $x_1 = x_2 = 40 - 1 = 39$ and $x_3 = 20 - 1 = 19$. All conditions hold, so we merely have to check that we are in fact allowed to select this point – which we are, as $a = 10 < x_3 < x_2 = x_1 < 50$.

[Note: If you didn't come up with the idea of $\sum \lambda \cdot (x_i + 1)$ what then? Eliminate $\lambda$ (which again must be nonzero, we cannot have $4/(x_1 + 1) = 0$) and then eliminate two of the $x_i$, for example as follows: Put FOCs #1 and #2 together for $\frac{4}{x_1+1} = \frac{4}{x_2+1}$ so $x_1 = x_2$. Eliminate $x_3$ as well: $x_3 + 1 = \frac{2}{\lambda} = \frac{1}{2} \cdot \frac{4}{\lambda} = \frac{1}{2}(x_1 + 1)$ and so $x_3 = \frac{1}{2}(x_1 - 1)$. Observing from the FOCs again that $\lambda$ cannot be zero, so that $x_1 + x_2 + x_3 = 97$, we insert for $x_2 = x_1$ and $x_3 = \frac{1}{2}(x_1 - 1)$ to get $97 = x_1 + x_1 + \frac{1}{2}(x_1 - 1)$ and $194 + 1 = 5x_1$ so that $x_1 = 39$. Then $x_2 = x_1 = 39$ and $x_3 = \frac{1}{2}(39 - 1) = 19$.]

(d) Reducing $M$ by $\frac{1}{2}$ will *reduce* the optimal value $V$ by $\approx \frac{1}{2}\frac{\partial V}{\partial M} = \frac{1}{2}\lambda = \frac{1}{2} \cdot \frac{1}{10} = 0.05$. (Reduction, so $\Delta V \approx -0.05$, but asking "how much" the change is could be answered in absolute value terms.)

(e) Since the optimum was $x_3 = 19$ when $a = 10$, we would suspect that the constraint $x_3 \geq 20$ will bind, but none of the other monthly constraints.

- For the first bullet item, we can actually start working out of this guess: we are asked to find a point satisfying the conditions, and if we have found one – even by checking first the case that happens to be right – we have answered the question, we do not need to search for more.

- For the second bullet item, we can do one out of two:
  - Either we can take note that by the extreme value theorem, a solution does exist; we can then find *all* points satisfying the Kuhn–Tucker conditions – with or without that "guess" holding true! – and compare them. For this particular problem there will be only one, but we need to do the work to show that.
  - Or, we have sufficient conditions: the concavity of the Lagrangian with the numbers for the multipliers inserted. This problem is particularly nice (a so-called *concave program*): The objective function is concave, the constraints are linear, so the Lagrangian will be concave no matter what the multipliers are.

    Thus, if we can find a point that satisfies the Kuhn–Tucker conditions – even if that was by checking the right case first – we can point at sufficient conditions and conclude that yes we have solved the problem.

Having pointed out that sufficient conditions will apply, we only need to find a point, and we can choose to first check the case we suspect: Try a minimal $x_3 = 20$, and so $x_1 + x_2 = 77$. The two first FOC are as before,

$$4 = \lambda(x_1 + 1) \qquad\qquad 4 = \lambda(x_2 + 1)$$

and lead to $x_1 = x_2$, which has to be $= \frac{77}{2}$ since $x_1 = x_2 = 77$. And $\frac{77}{2} \in (20, 50)$, that is OK – and $\mu_1 = \mu_2 = \gamma_1 = \gamma_2 = 0$. This gives a positive $\lambda$ too; below we shall need that $\lambda = \frac{4}{\frac{77}{2}+1} = \frac{8}{79}$.

We need to check the first-order condition wrt. $x_3$. It says

$$\frac{2}{x_3 + 1} - \lambda + \gamma_3 - \mu_3 = 0$$

and with $x_3 = a$, $\mu_3 = 0$. With $x_3 + 1 = a + 1 = 21$, this gives $\gamma_3 = \lambda - \frac{2}{21}$. The final condition to check is that $\gamma_3 \geq 0$: But $\gamma_3 = \frac{8}{79} - \frac{2}{21}$, and we are given that this is positive.

## Problem 3

(a) With $u = e^{e^t}$ then

$$du = e^{e^t} e^t dt = u \ln u \; dt \qquad \text{which yields} \quad dt = \frac{du}{u \ln u}$$

which we insert:

$$\int \frac{e^{2t}}{e^{e^t}} dt = \int \frac{(\ln u)^2}{u} \frac{du}{u \ln u} = \int \frac{\ln u}{u^2} du = -\frac{\ln u}{u} + \int \frac{1}{u^2} du = -\frac{1}{u}(\ln u + 1) + C$$

using integration by parts. Since $\ln u = e^t$, substituting back yields $C - \frac{e^t+1}{e^{e^t}}$ as should.

(b) This is a linear differential equation of the form $\dot{x} + ax = b(t)$ with $a = 1$ (with an antiderivative $A(t) = t$) and $e^{A(t)}b(t) = e^t \frac{e^t}{e^{e^t}} = \frac{e^{2t}}{e^{e^t}}$ which is the integrand in (a). Thus

$$x(t) = e^{-A(t)}\left[C - \frac{e^{-t}+1}{e^{e^t}}\right] = Ce^{-t} - \frac{e^{-t}+1}{e^{t+e^t}}$$

or some equivalent form.

Note: The the "$C$" in the formula is there to take care of the constant of integration so that you can replace $\int b(t)e^{A(t)}dt$ by "an antiderivative" (i.e. without constant). If you do not catch that point, rename constants (e.g. a $Q$ in the formula) to safeguard against ending up wrongfully cancelling constants against each other; the difference between two general constants, is a general constant and not "$C - C$".

## Problem 4

(a)

$$\begin{aligned}
dR &= e^H\left(H'_K\,dK + H'_L\,dL + H'_z\,dz\right)\\
&= e^H\left(\frac{a-z}{K}dK + \frac{a-z}{L}\,dL - (\ln K + \ln L)\,dz\right)\\
dS &= (a-z)\,dR - R\,dz\\
&= (a-z)e^H\left(\frac{a-z}{K}\,dK + \frac{a-z}{L}\,dL - (\ln K + \ln L)\,dz\right) - e^H dz
\end{aligned}$$

(b) Note first that

$$\begin{aligned}
dG &= \frac{a}{K}\,dK + \frac{b}{L}\,dL - 2(\ln z + 1)\,dz\\
dH &= \frac{a-z}{K}\,dK + \frac{a-z}{L}\,dL - (\ln K + \ln L)\,dz
\end{aligned}$$

And differentiating the first and second equations:

$$\begin{aligned}
ae^G dG + (a-z)e^H dH - e^H dz &= 2a\,dK\\
be^G dG + (b-z)e^H dH - e^H dz &= 3w\,dL + 3L\,dw
\end{aligned}$$

(c) We want to know how $K$ changes when we only change $w$ (not $z$, but $L$ is a dependent variable and will change as well). Thus $dz = 0$, moreover $K = 1/2$ and $L = 2$, and the equations simplifies to

$$\begin{aligned}
dG &= \frac{a}{K}\,dK + \frac{b}{L}\,dL = 2a\,dK + \frac{b}{2}\,dL\\
dH &= \frac{a-z}{K}\,dK + \frac{a-z}{L}\,dL = 2(a-z^*)\,dK + \frac{a-z^*}{2}\,dL
\end{aligned}$$

Note also that given $K$ and $L$ we can compute

$$
\begin{aligned}
G &= (b-a)\ln 2 - 2z^* \ln z^* \\
H &= (b-a)\ln 2
\end{aligned}
$$

and we also have

$$
\begin{aligned}
ae^G dG + (a-z)e^H dH &= 2a\, dK \\
be^G dG + (b-z)e^H dH &= 3w\, dL + 3L\, dw
\end{aligned}
$$

inserting for $dG$ and $dH$

$$
\begin{aligned}
ae^G\left(2a\, dK + \frac{b}{2}\, dL\right) + (a-z^*)e^H\left(2(a-z^*)\, dK + \frac{a-z^*}{2}\, dL\right) &= 2a\, dK \\
be^G\left(2a\, dK + \frac{b}{2}\, dL\right) + (b-z^*)e^H\left(2(a-z^*)\, dK + \frac{a-z^*}{2}\, dL\right) &= 3w\, dL + 3L\, dw
\end{aligned}
$$

We introduce coefficients $A$, $B$, $C$ and $D$:

$$
\begin{aligned}
A\, dK + B\, dL &= 0 \\
C\, dK + D\, dL &= 3L\, dw
\end{aligned}
$$

where

$$
\begin{aligned}
A &= 2a^2 e^G + 2(a-z^*)^2 e^H - 2a \\
B &= \frac{ab}{2}e^G + \frac{(a-z^*)^2}{2}e^G \\
C &= 2abe^G + 2(b-z^*)(a-z^*)e^H \\
D &= \frac{b^2}{2}e^G + \frac{(b-z^*)(a-z^*)}{2}e^G - 3w^*
\end{aligned}
$$

We can eliminate $dL$. The following way to eliminate, will give the answer on a form similar to what you get by Cramér's rule: Scale the first equation by $D$ and the second by $B$, so that both $dL$ coefficients are $BD$. Subtract equations to get

$$
(AD - BC)\, dK = -3B \cdot L\, dw \qquad \text{which yields} \qquad \frac{\partial K}{\partial w} = \frac{-3B \cdot L}{AD - BC}
$$

where the coefficients $A$, $B$, $C$ and $D$ are given as functions of $w^*$ and $z^*$ (and also depend on the constants $a$ and $b$).

Note: this is the situation where one is not expected to consider whether the divisor (the determinant $AD - BC$) is nonzero.