ECON3120/4120: The autumn 2022 exam solved

No formal guidelines were produced prior to grading. In case of appeals, an appeals committee should consult the Department. This document gives a suggested solution and some comments.

Problem 1:

(a) In order:

$$\mathbf{uv} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} \text{ is undefined}$$

($\mathbf{u}$ has one column and that number doesn't match the number of rows of $\mathbf{v}$)

$$\mathbf{u}'\mathbf{v} = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = -6$$

$$\mathbf{vw}' = \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & -2 & 0 \end{pmatrix}$$

$$\mathbf{A}_h\mathbf{u} = \begin{pmatrix} 1 & 0 & 2 \\ -h & 0 & -h \\ 2h & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1+0+2 \\ 0+0+0 \\ -2h+0-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2h-1 \end{pmatrix}$$

$$\mathbf{A}_h\mathbf{v} = \begin{pmatrix} 1 & 0 & 2 \\ -h & 0 & -h \\ 2h & 1 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 4-4 \\ -4h+2h \\ 8h+2+2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2h \\ 8h+4 \end{pmatrix}$$

$$\mathbf{A}_h\mathbf{w} = \begin{pmatrix} 1 & 0 & 2 \\ -h & 0 & -h \\ 2h & 1 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

We note that that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ make up the columns of $\mathbf{M}$, so we can use the three latter vectors for the first, second and third column of $\mathbf{A}_h\mathbf{M}$. However you do it, the result is

$$\mathbf{A}_h\mathbf{M} = \begin{pmatrix} 1 & 0 & 2 \\ -h & 0 & -h \\ 2h & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 4 & 0 \\ 0 & 2 & 1 \\ 1 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2h & 0 \\ -2h-1 & 8h+4 & 1 \end{pmatrix}$$

(b) Here we note from the last question in a) that $\mathbf{A}_h\mathbf{M} = \mathbf{I}$ if $h = -1/2$. Thus it is natural to choose $h = -1/2$, as $\mathbf{M}$ is the inverse in that case.

(c) The easiest is likely cofactor expansion along the second column – although in that case, remember the -1 to get $(-1)\cdot 1 \cdot \begin{vmatrix} 1 & 2 \\ -h & -h \end{vmatrix} = -(-h+2h) = -h$. But any row or column would do, for example the first row:

$$\begin{vmatrix} 1 & 0 & 2 \\ -h & 0 & -h \\ 2h & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 0 & -h \\ 1 & -1 \end{vmatrix} - 0 + 2\begin{vmatrix} -h & 0 \\ 2h & 1 \end{vmatrix} = +h - 2h = -h.$$

As there is no inverse if an only if the determinant is zero, there is no inverse only when $h \neq 0$.

(d) We know that $\mathbf{A}_h$ has an inverse when $h \neq 0$, in that case each of the equations will have a solution (and only one each). The only remaining case where we do not yet know if there is a solution, is when $h = 0$. We note that for $h = 0$, $\mathbf{A}_h$ reduces to

$$\mathbf{A}_0 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

and for any vector $\mathbf{x}$, then

$$\mathbf{A}_0\mathbf{x} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 - 2x_3 \\ 0 \\ x_2 - x_3 \end{pmatrix}$$

Thus the second element in $\mathbf{A}_0\mathbf{x}$ is zero, so neither $\mathbf{A}_0\mathbf{x} = \mathbf{v}$ nor $\mathbf{A}_0\mathbf{x} = \mathbf{w}$ can hold true. It remains to check $\mathbf{A}_0\mathbf{x} = \mathbf{u}$, where the system becomes

$$\mathbf{A}_0\mathbf{x} = \begin{pmatrix} x_1 - 2x_3 \\ 0 \\ x_2 - x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

For any x_3, let $x_1 = -1 - 2x_3$ and $x_2 = 1 + x_3$ and we have a solution. Thus there is an infinite number of solutions to $\mathbf{A}_0\mathbf{x} = \mathbf{u}$, one for each choice of x_3.

(Remark: It so happens for this particular system that *any single* variable can be chosen freely; choose instead x_1 and the first equation determines x_3 and the last then x_2; if instead x_2 is chosen, $x_3 = x_2 - 1$ from the third and $x_1 = 2x_3 - 1 = 2(x_2 - 1) - 1$ from the first equation.)

Problem 2

(a) The equation system is

$$\begin{aligned} y\ln(t+xs)+e^{xy^2} &= 1 \\ y^x+ty+x+s^2 &= 27 \end{aligned}$$

and we differentiate it (term by term or variable by variable, your choice) as

$$\begin{aligned} \ln(t+xs)\,dy+\frac{y}{t+xs}(\,dt+x\,ds+s\,dx)+e^{xy^2}(y^2\,dx+2yx\,dy) &= 0 \\ y^x(\ln y\,dx+\frac{x}{y}\,dy)+y\,dt+t\,dy+\ dx+2s\,ds &= 0 \end{aligned}$$

(b) We were given the point $(s,t,x,y)=(5,1,0,1)$, and we want to approximate $y(\frac{497}{100},1)$; that is, we change s by $ds=3/100$ and t by $dt=0$ and calculate the approximation $y(\frac{497}{100},1)\approx y(5,1)-\frac{3}{100}\frac{\partial y}{\partial s}|_{(s,t,x,y)=(5,1,0,1)}=1-\frac{3}{100}\frac{\partial y}{\partial s}|_{(s,t,x,y)=(5,1,0,1)}$. To calculate the partial derivative at the point, insert (the "0 without any d" are because $dt=0$):

$$\begin{aligned} \ln(1+0s)y+\frac{1}{1+0s}(0+0\,ds+5\,dx)+e^0(1\,dx+0\,dy) &= 0 \\ e^0((\ln 1)\ dx+\frac{0}{y}\,dy)+0+1\,dy+\ dx+10\,ds &= 0 \end{aligned}$$

Which simplifies to

$$\begin{aligned} 6dx &= 0 \\ 10ds+dx+dy &= 0 \end{aligned}$$

and so $dy=-10\,ds$ and $\partial y/\partial s=-10$. Inserting, we get the approximation $1-\frac{3}{100}\cdot(-10)=1.3$.

Problem 3

(a) Start with the indefinite integral $\int z^{R-1}(1+R\ln z)dz$. Let $w' = z^{R-1}$ and $v = 1+R\ln z$ in order to differentiate the latter: $v' = R/z$. For an antiderivative w of z^{R-1}, choose z^R/R. We get

$$\int z^{R-1}(1+R\ln z)dz = \frac{1}{R}z^R(1+R\ln z) - \frac{1}{R}\int Rz^{R-1}dz$$
$$= \frac{1}{R}z^R(1+R\ln z) - \frac{1}{R}z^R + C = z^R\ln z + C$$

Then for the definite integral: $\left[z^R\ln z + C\right]_{z=1}^{z=t} = t^R\ln t + C - C = t^R\ln t$ as should.

(b) With $u = R\ln z$, $du = \frac{R}{z}dz$ and so $dz = \frac{z}{R}du$. We need to get rid of z; as $\ln z = u/R$, $z = e^{u/R}$ and $dz = \frac{e^{u/R}}{R}du$. We also need $z^{R-1} = (e^{u/R})^{R-1} = e^{u(R-1)R/R}$. Inserting into the indefinite integral $\int z^{R-1}(1+R\ln z)dz$ yields

$$\int z^{R-1}(1+R\ln z)dz = \int (1+u)e^{u\frac{R-1}{R}}\frac{e^{\frac{u}{R}}}{R}du = \frac{1}{R}\int e^u(1+u)du$$

since $u\frac{R-1}{R} + \frac{u}{R} = u$.

The next step would be integration by parts, differentiating the $(1+u)$ and antidifferentiating e^u.

(c) $\dot{x} = (1-e^{-x})\frac{1-\ln t}{t^2}$ is a separable differential equation. We can separate by dividing by $1-e^{-x}$ provided that this is nonzero; it is zero if and only if $x = 0$, and so already we have the first of the particular solutions asked for: the constant solution $x(t) \equiv 0$.

For the other particular solution, we separate into $\frac{\dot{x}}{1-e^{-x}} = \frac{1-\ln t}{t^2}$ to obtain

$$\int \frac{dx}{1-e^{-x}} = \int \frac{1-\ln t}{t^2}dt$$

For the right hand side, use (a) with $R = -1$ to get $t^{-1}\ln t + C$.

For the left hand side, $\displaystyle\int \frac{dx}{1-e^{-x}} = \int \frac{e^x}{e^x-1}dx$ by the hint, and we substitute $u = e^x - 1$, so $du = e^x dx$: $\int \frac{du}{u}$. An antiderivative of u^{-1} is $\ln|u|$, substitute back to get $\ln|e^x-1|$ and put this equal to $t^{-1}\ln t + C$:

$$\ln|e^x - 1| = C + \frac{\ln t}{t}$$

At $t = 1$ we have $x = \ln 2$, so that $e^{x(1)} - 1 = 2 - 1 = 1$; because this is positive, we want $\ln(e^x - 1)$ and not $\ln(1-e^x)$. The left hand side becomes $\ln 1 = 0$ and the right hand side becomes $C + 0/1 = C$, so $C = 0$. Finally we solve $\ln(e^x - 1) = \frac{\ln t}{t}$:

$$e^x - 1 = e^{(\ln t)/t} = (e^{\ln t})^{1/t} = t^{1/t} \quad\iff\quad e^x = 1 + t^{1/t} \quad\iff\quad x = \ln(1+t^{1/t})$$

Problem 4

(a) First we note that $px + qy \leq m$, and $x \geq 0$; $y \geq 0$ defines a closed and bounded set (non-empty too, although that is not the main focus in this course). Since the function is continuous, there is a maximum by the extreme value theorem. We also know the function is non-decreasing in x and in y, so if $px + qy < m$ we may increase both x and y and hence get no lower utility. Thus an interior point cannot be better than one on the budget line.

(b) Euler's formula gives

$$xu'_x + yu'_y = d \cdot u$$

Now, at an interior maximum $u'_x = u'_y = 0$, and since $d > 0$, it follows that

$$u = \frac{1}{d}\left(xu'_x + yu'_y\right) = 0$$

(c) Now

$$u(x, y) = ax + by + c$$

To test for homogeneity:

$$u(tx, ty) = atx + bty + c \text{ and this equals } tu(x, y) \text{ if and only if } c = 0$$

Then for $c = 0$ it is homogeneous – and of degree 1.*

The function is homothetic if it is a monotone transformation of a homogenous function. *Adding a constant* makes for a strictly increasing transformation (as $z + c$ is strictly increasing in z), so

$$ax + by + c$$

is a monotone transformation of an homogenous function $ax + by$. Thus the function is homogenous of degree 1 if $c = 0$, and homothetic for all values of c.

(d) The Lagrangian is

$$L(x, y) = ax + by + c + \alpha x + \beta y - \lambda(px + qy - m)$$

- The Kuhn–Tucker conditions become:

$$a + \alpha - \lambda p = 0$$
$$b + \beta - \lambda q = 0$$
$$\alpha \geq 0; \quad \text{and with } \alpha = 0 \text{ if } x > 0$$
$$\beta \geq 0; \quad \text{and with } \beta = 0 \text{ if } y > 0$$
$$\lambda \geq 0; \quad \text{and with } \lambda = 0 \text{ if } px + qy < m$$

*This would suffice, but if you want more detailed that a nonzero c rules out homogeneity – of any degree – then you can argue as follows: $u(tx, ty) = t(ax + by + c) - tc + c = tu(x, y) + (1 - t)c$; This being $= t^d u(x, y)$ if and only if $t^d u(x, y) - tu(x, y) = (1 - t)c$ identically. Since $(1 - t)c$ does not depend on (x, y), then neither can $(t^d - t)u(x, y)$, and the only way to kill the (x, y) dependence is to have $d = 1$. Then $(1 - t)c$ must be $= 0$ for all $t > 0$, and so $c = 0$ is necessary.

- Suppose for contradiction that $xy \neq 0$. Then $x > 0$ and $y > 0$ so $\alpha = 0$ and $\beta = 0$. The Kuhn–Tucker conditions reduce to

  $$a = \lambda p$$
  $$b = \lambda q$$

  Solving for λ and rearranging:

  $$\lambda = \frac{a}{p} = \frac{b}{q}$$

  But we are given that $\frac{a}{p} > \frac{b}{q}$. As the assumption $xy \neq 0$ violates a conditions stated in the problem, it cannot be true; as $xy \neq 0$ must be false that means that $xy = 0$ is true.

(e) Note first that

$$a + \alpha = \lambda p$$
$$b + \beta = \lambda q$$

Since α and β are non-negative and $a > 0$ and $b > 0$, the left hand sides are strictly positive. Thus it is impossible to have $\lambda = 0$ and thus impossible to have $px + qy < m$. As we know from (d) that $xy = 0$, that leaves us with two possible solutions:

$$x = 0 \text{ and } y = \frac{m}{q} \text{ and } \beta = 0$$
$$y = 0 \text{ and } x = \frac{m}{p} \text{ and } \alpha = 0$$

To decide which of this is the solution (and we know from the extreme value theorem that there is one!), we have one condition at hand, namely that $\dfrac{a}{b} > \dfrac{p}{q}$. One way to settle it is to note that

$$\lambda = \frac{a + \alpha}{p} = \frac{b + \beta}{q} \quad \text{and the latter is equivalent to } \frac{a + \alpha}{b + \beta} = \frac{p}{q}.$$

One of α and β is zero; the other is strictly positive (we cannot have both zero, it would violate $a/b > p/q$). Check the two possible cases in turn:

Case $\alpha > \beta = 0$: Then $\dfrac{p}{q} = \dfrac{a + \alpha}{b}$ which is $= \dfrac{a}{b} + \dfrac{\alpha}{b}$ which is $> \dfrac{a}{b}$, leading to $\dfrac{p}{q} > \dfrac{a}{b}$ which is impossible.

Having eliminated this case, it must be the other (as by the extreme value theorem we know we have a solution, and nothing else is possible!) But if we were to check that case, it would yield:

Case $\beta > \alpha = 0$: Then $\dfrac{p}{q} = \dfrac{a}{b + \beta}$ which is $< \dfrac{a}{b}$ (we divide by something bigger). Possible indeed!

So the latter case with $\alpha = 0 < \beta$ solves the problem, and the solution is found at $(x, y) = (\frac{m}{p}, 0)$.