ECON3120/4120 Mathematics 2

December 18th 2023, 1500–1900 (4 hrs). (English only.) There are 4 problems to be solved.
Support material: "Rules and formulas" attachment.

For the entire problem set:

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Suggested weighting: letter-enumerated problems count equal.

Problem 1 Take for granted that the equation system

$$\begin{aligned}(4-x)^2 + yx^{3/2} - t &= 5 \\ 16x + 3ye^{x-1} &= 24\end{aligned} \tag{S}$$

determines continuously differentiable functions $x = x(t)$ and $y = y(t)$ around the point where $(t, x, y) = (\frac{20}{3}, 1, \frac{8}{3})$.

(a) Differentiate the equation system.

(b) Use the differentiated system to find an approximate value for $x(7)$.

Problem 2 For each real constant w, define $\mathbf{M}_w = \begin{pmatrix} 2 & -5 & 0 \\ 1 & 0 & -3 \\ 0 & 4 & w \end{pmatrix}$ and $\mathbf{b}_w = \begin{pmatrix} 0 \\ 4 \\ w \end{pmatrix}$.

Let $\mathbf{S} = \mathbf{M}_w'\mathbf{M}_w$, $\mathbf{T} = \mathbf{b}_w\mathbf{b}_w'$, $\mathbf{U} = (\mathbf{b}_w)^2$ and $\mathbf{V} = \mathbf{M}_w\mathbf{b}_w$, *provided they are well-defined.*

(a) For each of the matrix products $\mathbf{S}$, $\mathbf{T}$, $\mathbf{U}$, $\mathbf{V}$:

- Calculate both the last row and the last column – or explain why the product does not exist. (Answers might depend on w.)
- When that last row is the transpose of the last column (and for at least one product it is): Could we tell that before multiplying out – or is that "a mere coincidence" by the particular elements of $\mathbf{M}_w$ and/or $\mathbf{b}_w$?

(b) For each of the matrices $\mathbf{M}_w$ and $\mathbf{S}$ and $\mathbf{V}$:
Calculate the determinant or point out why the determinant is not well-defined.

(c) For what value(s) of w will the equation system $\mathbf{M}_w\mathbf{x} = \mathbf{b}_w$ have:
No solution? Precisely one solution? Several solutions?

(d) Let $w = 0$. *Use Cramér's rule* to show that $x_2 = 0$. (There is no score for any other method.)

Problem 3

(a) Show that

$$\int \frac{e}{(e-x)x}\,dx = C - \ln\left|\frac{1}{e} - \frac{1}{x}\right|$$

(b) Does the integral $\displaystyle\int_{e/2}^{2e} \frac{e}{(e-x)x}\,dx$ converge?

(c) *Use integration by parts* to find constants A and B such that

$$\int_1^t 2s\ln(s^3e^4)\,ds = A\cdot(t^2-1) + B\cdot t^2\ln t$$

(d) Consider the differential equation

$$\dot{x} = (e-x)x\cdot t\ln(t^3e^4)$$

Find the following *two* particular solutions: The one such that $x(1) = e$, and the one such that $x(1) = e/2$.

You are free to express your answer(s) in terms of the symbols «A» and «B» without inserting the actual numbers from part (c).

Problem 4 Let $F(x, y) = x + y - xy - x^2 - y^2$.

(a) Is F homogeneous? *Hint:* Euler's theorem.

Consider first the problems

$$\max F(x,y) \quad \text{subject to} \quad 12x + 6y = 11 \tag{L}$$
$$\max F(x,y) \quad \text{subject to} \quad 12x + 6y \geq 11 \tag{K}$$

($12x+6y \geq 11$ is equivalent to $11-12x-6y \leq 0$ if you prefer the inequality in that direction.)

(b) Consider the point $(\hat{x}, \hat{y}) = (\frac{3}{4}, \frac{1}{3})$.

- Verify that $(\hat{x}, \hat{y})$ satisfies the Lagrange conditions associated with problem (L).
- Does $(\hat{x}, \hat{y})$ satisfy the Kuhn–Tucker conditions associated with problem (K)?

In the following, you can use without proof the fact that $(\hat{x}, \hat{y})$ is the *only* point that satisfies the Lagrange conditions associated with problem (L).

Consider now the following optimization problem, with one more constraint than (K):

$$\max F(x,y) \quad \text{subject to} \quad 12x + 6y \geq 11 \quad \text{and} \quad x + y \leq 1 \tag{P}$$

(c) Show that if the Kuhn–Tucker conditions associated with (P) hold at an admissible point (x, y) then we cannot have $x + y < 1$.

(d) It is a fact that there is one unique point (x^*, y^*) that satisfies the Kuhn–Tucker conditions, and that both the Lagrange multipliers are > 0.
Show that (x^*, y^*) solves problem (P).

(End of problem set. Attachment: Rules and formulas.)

Attachment: Rules and formulas

A. Exponentials and logarithms For base numbers $b > 0$ with $b \neq 1$:

(A1) $b^{-x} = 1/b^x$ $\quad b^{x\pm y} = b^x \cdot b^{\pm y}$ $\quad b^{x+yz} = b^x \cdot (b^y)^z$

(A2) for $x > 0, y > 0$: $\quad b^{\log_b x} = x$ $\quad \log_b (x \cdot y^z) = \log_b x + z \log_b y$ $\quad \log_b x = \dfrac{\log_c x}{\log_c b}$

We write ln for the *natural* logarithm $\log_e$ where $e = \lim_{n\to+\infty} (1 + \frac{1}{n})^n \approx 2.718281828$.

B. Limits Notational convention in this course: when $\lim_{x\to a}$, x is never equal to a. For example, in the definition $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$, we let $h \to 0$ without touching zero. For a limit to *exist* (it «converges»), it must be *finite,* but we write e.g. $\lim_{x\to 0} x^{-2} = +\infty$ («diverges» to $+\infty$, not converges). Limits that diverge but not to $\pm\infty$ are not significant in Math 2 (example: $\lim_{n\to+\infty}(-1)^n$, n runs through the natural numbers only).

Rules If $\ell = \lim_{x\to a} f(x)$ and $m = \lim_{x\to a} g(x)$ both *exist* (implying: are *finite*):

(B1) $\lim_{x\to a} \big(f(x) \pm g(x)\big) = \ell \pm m, \quad \lim_{x\to a} \big(f(x)g(x)\big) = \ell m, \quad \lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\ell}{m}$ if $m \neq 0$

Same validity if the «$x \to a$» are replaced by $x \to a^+$ or $x \to a^-$ or $x \to -\infty$ or $x \to +\infty$. *When ℓ exists and m does not,* the first formula holds in the sense that ℓ+[does not exist] does not exist, $\ell \pm (+\infty) = \ell \pm \infty$ etc.; for the second formula, we can write $\ell \cdot (+\infty) = \infty \cdot \operatorname{sign} \ell$ *if* $\ell \neq 0$ but this inference is invalid if $\ell = 0$. For the third, we have $\ell/(\pm\infty) = 0$.

Continuity A function is continuous at some a in its domain, if $\lim_{x\to a} f(x)$ exists and equals $f(\lim_{x\to a} x) = f(a)$, i.e. limits can be computed inside the function. It is *continuous* if it is continuous at every a in its domain. Compositions of continuous functions are continuous. *Note,* in Math 2 one does not need to argue that a particular function is continuous *where it is defined* – as long as one does not make incorrect claims.

l'Hôpital's rule If the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ *are both zero,* or *both diverge to infinity:*

(B2) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$ (finite or infinite; the former diverges if the latter diverges)

Same validity if the «$x \to a$» are replaced by $x \to a^+$ or $x \to a^-$ or $x \to -\infty$ or $x \to +\infty$. *You must justify the validity* when using l'Hôpital's rule; e.g. as the overbraces in the following significant examples: For $p > 0$ and $q > 0$, using continuity of t^p and the differentiation rules:

(B3) $$\lim_{x\to+\infty} \frac{x^p}{e^{qx}} = \Big(\overbrace{\lim_{x\to+\infty} \frac{x}{e^{qx/p}}}^{=«+\infty/+\infty»}\Big)^p = \Big(\lim_{x\to+\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^{qx/p}}\Big)^p = \Big(\lim_{x\to+\infty} \frac{1}{\frac{q}{p}e^{qx/p}}\Big)^p = 0^p = 0$$

(B4) $$\lim_{x\to+\infty} \frac{(\ln x)^p}{x^q} = \Big(\overbrace{\lim_{x\to+\infty} \frac{\ln x}{x^{q/p}}}^{=«+\infty/+\infty»}\Big)^p = \Big(\lim_{x\to+\infty} \frac{1/x}{\frac{q}{p}x^{q/p-1}}\Big)^p = \Big(\frac{p}{q} \lim_{x\to+\infty} x^{-q/p}\Big)^p = 0^p = 0$$

(B5) $$\lim_{x\to 0^+} x^q \,|\ln x|^p = \Big|\overbrace{\lim_{x\to 0^+} \frac{\ln x}{x^{-q/p}}}^{=«-\infty/+\infty»}\Big|^p = \Big|\lim_{x\to 0^+} \frac{1/x}{-\frac{q}{p}x^{-1-q/p}}\Big|^p = \Big|\frac{p}{q} \lim_{x\to 0^+} x^{q/p}\Big|^p = 0^p = 0$$

(B6) $\lim f(x) = e^{\lim \ln f(x)}$ if $f(x) > 0$; in particular useful if $\lim f(x)$ is «1^∞», «∞^0», «0^0».

C. Derivatives, differentials, elasticities

Provided differentiability and no division by 0:

(C1) $$\frac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x), \qquad \frac{d}{dx} g(f(x)) = g'(f(x))f'(x)$$

(C2) $$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x) \qquad \frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

(C3) $$\frac{d}{dx}x^r = rx^{r-1}, \qquad \frac{d}{dx}|x| = \frac{x}{|x|} = \frac{|x|}{x}, \qquad \frac{d}{dx}e^x = e^x, \qquad \frac{d}{dx}\ln|x| = \frac{1}{x}$$

For b^x, respectively $\log_b x$: Write as $e^{x \ln b}$ resp. $\frac{\ln x}{\ln b}$. If $f(x) > 0$, then $f'(x) = f(x)\frac{d}{dx}\ln f(x)$. For inverse functions: $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$.

Partial derivatives $\frac{\partial f}{\partial x_i}$: similar rules.

The differential: if $z = f(x_1, \dots, x_n)$, we define the differential dz to be: $\frac{\partial f}{\partial x_1}(x_1, \dots, x_n)\, dx_1 + \cdots + \frac{\partial f}{\partial x_n}(x_1, \dots, x_n)\, dx_n$. Differentials obey rules similar to derivatives.

Elasticities: $\mathrm{El}_x f(x) = \frac{x}{f(x)} f'(x)$ for $f(x) \neq 0$. Can be written as $\mathrm{El}_x f(x) = \frac{d \ln |f(x)|}{d \ln |x|}$ (which equals $\frac{d \ln f(x)}{d \ln x}$ if $f > 0$ and $x > 0$). Rules, assuming functions and arguments positive:

(C4) $$\mathrm{El}_x\Big(f(x) \cdot g(x)^r\Big) = \frac{d \ln f(x) + d\big(r \ln g(x)\big)}{d \ln x} = \mathrm{El}_x f(x) + r\mathrm{El}_x g(x)$$

(C5) $$\mathrm{El}_x\Big(f(x) \cdot g(x)^{h(x)}\Big) = \mathrm{El}_x f(x) + h(x) \cdot \big[\mathrm{El}_x g(x) + \ln g(x) \cdot \mathrm{El}_x h(x)\big]$$

(C6) $$\mathrm{El}_x g(f(x)) = \frac{d \ln g(u)}{d \ln u}\Big|_{u=f(x)} \cdot \frac{d \ln f(x)}{d \ln x}$$

(C7) $$\mathrm{El}_x\Big(f(x) + g(x)\Big) = \frac{x(f'(x) + g'(x))}{f(x) + g(x)} = \frac{f(x)\mathrm{El}_x f(x) + g(x)\mathrm{El}_x g(x)}{f(x) + g(x)}$$

For functions of several variables, El_{x_i} denotes *partial* elasticity in this course.

Implicit derivatives If $(\mathbf{x}, z)$ satisfies $F(x_1, \dots, x_n, z) = C$, then $\sum_i F'_{x_i}(\mathbf{x}, z)\, dx_i + F'_z(\mathbf{x}, z)\, dz = 0$ and as long as $F'_z(\mathbf{x}, z) \neq 0$, the equation determines $z = g(\mathbf{x})$ with $\frac{\partial g}{\partial x_i} = -\frac{F'_{x_i}(\mathbf{x},z)}{F'_z(\mathbf{x},z)}$.
If two equations $F(\mathbf{x}, K, L) = C$ and $G(\mathbf{x}, K, L) = D$ determine continuously differentiable functions $K = K(\mathbf{x})$ and $L = L(\mathbf{x})$, then the following recipe gives their partial derivatives:

- Differentiate the equation system (i.e. calculate differentials). Obtain
$$F'_K(\mathbf{x}, K, L)\, dK + F'_L(\mathbf{x}, K, L)\, dL + \sum_i F'_{x_i}(\mathbf{x}, K, L)\, dx_i = 0$$
$$G'_K(\mathbf{x}, K, L)\, dK + G'_L(\mathbf{x}, K, L)\, dL + \sum_i G'_{x_i}(\mathbf{x}, K, L)\, dx_i = 0$$
- This is a linear equation system in dK and dL, when everything else is taken as constant. Solve it to obtain the following (you are not required to use matrix notation):

(C8) $$\begin{pmatrix} dK \\ dL \end{pmatrix} = -\mathbf{A}^{-1} \sum_i \begin{pmatrix} F'_{x_i}(\mathbf{x}, K, L) \\ G'_{x_i}(\mathbf{x}, K, L) \end{pmatrix} dx_i \quad \text{where } \mathbf{A} = \begin{pmatrix} F'_K(\mathbf{x}, K, L) & F'_L(\mathbf{x}, K, L) \\ G'_K(\mathbf{x}, K, L) & G'_L(\mathbf{x}, K, L) \end{pmatrix}$$

- This gives the forms $dK = \sum_i \kappa_i\, dx_i$ and $dL = \sum_i \lambda_i\, dx_i$. Then $\frac{\partial K}{\partial x_i} = \kappa_i$ and $\frac{\partial L}{\partial x_i} = \lambda_i$.

D. Optimization etc. Several of the following statements omit a requirement that the set S be «convex», as that is beyond Mathematics 2. (Convex subsets of $\mathbf{R}$ = the intervals.)

Some terminology: «open» resp. «closed» set: includes none resp. all of its boundary points. A «maximum» resp. «minimum» for f: an $\mathbf{x}^*$ (i.e. a *point*) such that for all $\mathbf{x}$ we have $f(\mathbf{x}) \leq f(\mathbf{x}^*)$ (resp. $\geq f(\mathbf{x}^*)$). The output $f(\mathbf{x}^*)$ is called the maximum/minimum *value.* E.g., the max/min for $f(x) = ax^2 + bx + c$ (if $a \neq 0$), is $x^* = \frac{-b}{2a}$; the max/min *value* is $c - \frac{b^2}{4a}$.

Two existence theorems: Let f be (defined and) continuous on the entire set S.

(D1) *The extreme value theorem:* If $S \subset \mathbf{R}^n$ is closed, bounded and nonempty, then the continuous function f has both a maximum and a minimum over S.

(D2) *The intermediate value theorem:* If $n = 1$ and $S = [a, b]$ (interval, endpoints contained), then the continuous function f attains every value between $f(a)$ and $f(b)$ at least once.

Convex and concave function of one variable: Let f be C^1, defined on an interval.
f is convex (respectively: concave) if f' is nondecreasing (resp. nonincreasing) everywhere.
If f is also C^2, then it is convex (respectively: concave) if $f'' \geq 0$ (resp. ≤ 0) everywhere.

Convex and concave function of two variables: Let f be C^2 on $S \subseteq \mathbf{R}^2$.
Let $h(x, y) = f''_{xx}(x, y) f''_{yy}(x, y) - \big(f''_{xy}(x, y)\big)^2$ (the so-called *Hessian determinant.*)

(D3) If and only if $h \geq 0$ and $f''_{xx} \geq 0$ and $f''_{yy} \geq 0$ on all of S, then f is convex on S

(D4) If and only if $h \geq 0$ and $f''_{xx} \leq 0$ and $f''_{yy} \leq 0$ on all of S, then f is concave on S

If $h(x, y) > 0$ at some given point, then $f''_{xx}(x, y)$ and $f''_{yy}(x, y)$ are nonzero and of same sign:

(D5) If $h(x, y) > 0$ and $f''_{xx}(x, y) > 0$ then f is strictly convex on some open set around (x, y)

(D6) If $h(x, y) > 0 > f''_{xx}(x, y)$ then f is strictly concave on some open set around (x, y)

Convex and concave function of n variables: The following are sufficient (but not necessary) for convexity/concavity. Let $\alpha \geq 0$ and $\beta \geq 0$ be constants.

(D7) If f and g are both convex (resp. concave), then $\alpha f + \beta g$ is convex (resp. concave)

Unconstrained optimization (i.e. on open set S). *First-order condition:* stationary point, i.e. $\partial f / \partial x_i$ equal zero at $\mathbf{x}^*$, all $i = 1, \ldots, n$. *Assuming* stationary point $\mathbf{x}^*$:

- *Global second-order condition:* If the function is convex (resp. concave), a stationary point $\mathbf{x}^*$ is a global min (resp. global max).
- *Local second-order condition for $n = 2$ variables:* Let (x^*, y^*) be a stationary point.
  If (D5) (resp. (D6)) holds at (x^*, y^*), it is a strict local min (resp. strict local max).
  If $h(x^*, y^*) < 0$, it is neither (a «saddle point»); if $h(x^*, y^*) = 0$, Math 2 cannot classify.
- 1 variable, $f'(x^*) = 0$: $f''(x^*) > 0 \Rightarrow$ strict local min. $f''(x^*) < 0 \Rightarrow$ strict local max.
- For 1 variable, the first-derivative test (a sign diagram is possibly useful):
  Increase x across x^*. If $f'(x)$ changes sign from negative to positive (resp. positive to negative), then x^* is local min (resp. local max). If furthermore x^* is the *only* change of sign of f' in the domain of f, the min (resp. max) is *global.*

Constrained optimization Problem type: $\max f(\mathbf{x})$ subject to constraints $g_j(\mathbf{x}) \leq b_j$ or $= b_j$ (m constraints, n variables) Form the Lagrangian $L(\mathbf{x}) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j (g_j(\mathbf{x}) - b_j)$.

Conditions – on the exam, they must be written out!

- Equality-only constraints, $m < n$: The Lagrange conditions for a point $\mathbf{x}^*$ to solve the problem, are that there exist numbers $\lambda_1, \ldots, \lambda_m$ such that $\mathbf{x}$ is a stationary point for L, and the constraints hold. $n + m$ equations for $\mathbf{x}$ and the λ_j.
  These conditions are the same for the problem with «min» in place of «max».
- Inequality-only constraints: The Kuhn–Tucker conditions for $\mathbf{x}^*$ to maximize, are that there exist *nonnegative* numbers $\lambda_1 \geq 0, \ldots, \lambda_m \geq 0$, such that $\mathbf{x}^*$ is a stationary point for L, and that if $g_j(\mathbf{x}^*) < b_j$ then $\lambda_j = 0$. That is:

  (D8) $$\frac{\partial L}{\partial x_i}(\mathbf{x}^*) = 0 \text{ for every } i\text{, and for every } j\text{:} \quad \lambda_j \geq 0 \text{ and if } g_j(\mathbf{x}^*) < b_j \text{ then } \lambda_j = 0$$

  Also the constraints must hold, and you are free to include them or not if asked for the «Kuhn–Tucker conditions». (Equivalent formulations are OK.)

Necessity/sufficiency etc.:

- In this course you can take the Lagrange / Kuhn–Tucker conditions as necessary.
- Sufficient conditions: Suppose $\mathbf{x}^*$ satisfies the Lagrange resp. Kuhn–Tucker conditions with numbers $\lambda_1, \ldots, \lambda_m$. Then $\mathbf{x}^*$ solves the maximization problem if:

  (D9) $\mathbf{x}^*$ maximizes L subject to to the constraints.
  This in particular holds if L is concave in $\mathbf{x}$.
- If condition (D9) can not be used, then you can compare values provided you have established existence (e.g. by the extreme value theorem (D1)).
- (Omitted at least in 2019: *Local* second-order condition for the Lagrange problem. (D10) Equation number advances by one for placeholder.)

Value functions, derivatives (envelope theorem), shadow prices. If f depends on $\mathbf{x}$ (choice variable) and $\mathbf{r}$ (exogenous), then – assuming maximum exists – the maximum value $\max_{\mathbf{x}} f(\mathbf{x}, \mathbf{r})$ is a function $V(\mathbf{r})$, and the (possibly) maximum (point) $\mathbf{x}^*$ depends on $\mathbf{r}$ as well.
The same applies when there are (possibly $\mathbf{r}$-dependent) *constraints.*

The envelope theorem: in the (possibly constrained) optimization problem, suppose f, the g_j and the b_j depend on $\mathbf{r}$. To the precision level of this course:

(D11) $$\frac{\partial V}{\partial r_i}(\mathbf{r}) = \frac{\partial f}{\partial r_i}(\mathbf{x}^*, \mathbf{r}) - \sum_{j=1}^{m} \lambda_j \Big(\frac{\partial g}{\partial r_i}(\mathbf{x}^*, \mathbf{r}) - \frac{\partial b}{\partial r_i}(\mathbf{r}) \Big)$$

The formula holds for stationary saddle points too, not just max/min. Special cases:

- Unconstrained: $m = 0$, remove the sum to get $\frac{\partial V}{\partial r_i}(\mathbf{r}) = \frac{\partial f}{\partial r_i}(\mathbf{x}^*, \mathbf{r})$.
- Unconstrained, one variable: It also holds for *endpoint* max/min.
- If there is no $\mathbf{r}$-dependence in f nor g_j nor b_j, then the value depends on the b_j constants, $V = V(\mathbf{b})$. Then $\frac{\partial V}{\partial b_j}(\mathbf{b}) = \lambda_j$ (the shadow price interpretation).

E. Integration. All functions on this page are of a single variable t, bounded and piecewise continuous – until specified otherwise in the Leibniz rule.

Terminology. If $F' = f$ on the domain of f, then F is an *antiderivative* of f. The *indefinite integral* $\int f(t)\,dt$ equals $F(t) + C$, i.e. the *general* antiderivative of f; here, C is an arbitrary constant. The *definite integral* $\int_a^b f(t)\,dt$ equals $F(b) - F(a)$.

Area. When $b \geq a$ and $f \geq 0$ on (a, b), the definite integral $\int_a^b f(t)dt$ equals the area delimited by the first axis and the graph of f between a and b. When f can take either sign, it equals the part of the area above the axis, minus the part of the area under the axis.

Rules. Derivatives rules (see (C1)–(C3)) can be applied in reverse. For α, β constant:

(E1) Sums and scalings: $$\int \Big(\alpha f(t) + \beta g(t)\Big)dt = \alpha \int f(t)dt + \beta \int g(t)dt$$

(E2) $$\textit{except:} \int \Big(f(t) - f(t)\Big)dt = \int 0\,dt = C \quad \text{(rather than zero)}$$

(E3) Integration by parts: $$\int f'(t)g(t)\,dt = f(t)g(t) - \int f(t)g'(t)\,dt$$

(E4) Integration by substitution: $$\int f'(u(t))u'(t)\,dt = \int f(u)\,du = F(u(t)) + C$$

(E5) ... in definite integrals: $$\int_a^b f'(u(t))u'(t)\,dt = \int_{u(a)}^{u(b)} f(u)\,du$$

You *will not* be asked to integrate «$\frac{t-\gamma}{(t-\alpha)(t-\beta)}$» when $\alpha \neq \beta$, but if it shows up due to your own calculations: rewrite into $\frac{\alpha-\gamma}{\alpha-\beta} \cdot \frac{1}{t-\alpha} - \frac{\beta-\gamma}{\alpha-\beta} \cdot \frac{1}{t-\beta}$. (When $\alpha = \beta$: write $\frac{t-\gamma}{(t-\alpha)^2}$ as $\frac{1}{t-\alpha} + \frac{\alpha-\gamma}{(t-\alpha)^2}$.)

Extension: improper integrals. The above assumes bounded integrand and bounded interval. Otherwise, the integral is defined as limits, provided they exist. When the integrand f is unbounded only near a and/or near $b > a$:

(E6) $$\int_a^b f(t)\,dt = \lim_{R \to a^+} \int_R^c f(t)\,dt + \lim_{S \to b^-} \int_c^S f(t)\,dt \quad \text{(both limits need to exist)}$$

If f unbounded only near $c \in (a, b)$, apply (E6) on each term $\int_a^c f(t)\,dt$ and $\int_c^b f(t)\,dt$.
For infinite intervals:

(E7) $$\int_{-\infty}^b f(t)\,dt = \lim_{R \to -\infty} \int_R^b f(t)\,dt, \qquad \int_a^{+\infty} f(t)\,dt = \lim_{S \to +\infty} \int_a^S f(t)\,dt$$

These rules/definitions can be combined by splitting into integrals with only one limit transition each. E.g. $\int_{-\infty}^{+\infty} f(t)\,dt = \int_{-\infty}^c f(t)\,dt + \int_c^{+\infty} f(t)\,dt$ for any c.

The Leibniz rule for differentiating integral expressions. Let f be a function of two variables (x, t) and note that for purposes of integration wrt. t, x is treated as constant. The formula

(E8) $$\frac{d}{dx} \int_{u(x)}^{v(x)} f(x, t)\,dt = f(x, v(x))v'(x) - f(x, u(x))u'(x) + \int_{u(x)}^{v(x)} f'_x(x, t)\,dt$$

is valid in Mathematics 2; also for improper integrals with infinity treated as constant.

F. Differential equations. A *particular* solution is a function that satisfies the differential equation. The *general* solution is the set of all particular solutions. You are expected to *verify* any proposed particular solution. To *find* solutions, you are expected to handle the following two types of (ordinary first-order) differential equations for the unknown $x = x(t)$:

Linear differential equations $\dot{x}(t) + a(t)x(t) = b(t)$. Let A be an antiderivative of a. Then $\frac{d}{dt}\big(e^{A(t)}x(t)\big) = (\dot{x}(t) + a(t)x(t))e^{A(t)}$, which $= b(t)e^{A(t)}$, and so $e^{A(t)}x(t) = \int b(t)e^{A(t)}\,dt$ and

$$x(t) = Ce^{-A(t)} + e^{-A(t)} \int b(t)e^{A(t)}\,dt \tag{F1}$$

Writing a constant C allows the integral to be any antiderivative, and so the right-hand side is the sum of *any given particular solution* $e^{-A(t)} \int b(t)e^{A(t)}\,dt$ and *the general solution* $Ce^{-A(t)}$ *of the corresponding homogeneous equation* (obtained by replacing b by the zero function).
For a particular solution: find C. Example with t_0 and $x(t_0) = x_0$ given: if $a \neq 0$ and b are constants, then $x(t) = (x_0 - b/a)e^{-a(t-t_0)} + b/a$ is of the form (F1) and satisfies $x(t_0) = x_0$.

Separable differential equations $\dot{x}(t) = f(t)g(x(t))$ (or, which can be rewritten that way). Note, g depends on x only. The general solution is found by (i) any zero z of g is a constant particular solution $x(t) \equiv z$, and (ii) for $g \neq 0$, separate into $\frac{dx}{g(x)} = f(t)\,dt$, integrate

$$\int \frac{1}{g(x)}\,dx = \int f(t)\,dt \qquad \text{which yields} \qquad H(x) = F(t) + C, \tag{F2}$$

solving the resulting algebraic equation for x and collecting the contributions from (i) and (ii).
For a particular solution satisfying $x(t_0) = x_0$: If $g(x_0) = 0$ (case (i)), the particular solution is $x(t) \equiv x_0$. Otherwise (case (ii)), find C as $H(x_0) - F(t_0)$ and solve for x.

G. Approximations. Taylor polynomials. Let f be a C^k function of a single variable. Its *kth order approximation* around $t = a$, is the kth order polynomial

$$p_{k,a}(t) = f(a) + f'(a)\cdot(t-a) + \frac{1}{2}f''(a)(t-a)^2 + \cdots + \frac{1}{k!}f^{(k)}(a)\cdot(t-a)^k \tag{G1}$$

where $f^{(j)}$ denotes the jth derivative $\left(\frac{d}{dt}\right)^j f$ and $j!$ denotes $j\cdot(j-1)\cdot\cdots\cdot 1$. If f is also C^{k+1}, then for each t there exists a c between t and a such that $f(t) - p_{k,a}(t) = f^{(k+1)}(c)\cdot\frac{1}{(k+1)!}(t-a)^{k+1}$.

In n variables: when $k = 2$, we have

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \sum_{i=1}^{n}(x_i - a_i)\frac{\partial f}{\partial x_i}(\mathbf{a}) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(x_i - a_i)(x_j - a_j)\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}) \tag{G2}$$

or in matrix notation, where the $\cdot$ denotes dot product:

$$f(\mathbf{x}) \approx f(\mathbf{a}) + \nabla f(\mathbf{a})\,\mathbf{z} + \tfrac{1}{2}\,\mathbf{z}\cdot\big(\mathbf{H_a}\,\mathbf{z}\big) \qquad \text{where } \mathbf{z} = \mathbf{x} - \mathbf{a} \text{ (column vector),} \tag{G3}$$

$\nabla f(\mathbf{a}) = \big(f_1'(\mathbf{a}), \dots, f_n'(\mathbf{a})\big)$ is the *gradient* (the row vector of first derivatives) at $\mathbf{a}$,

$\mathbf{H_a}$ is the *Hessian matrix at* $\mathbf{a}$*:* the $n \times n$ matrix with elements $h_{ij} = \dfrac{\partial^2 f}{\partial x_i\,\partial x_j}(\mathbf{a})$.

For $k = 1$, delete the quadratic terms to get $f(\mathbf{x}) \approx f(\mathbf{a}) + \sum_{i=1}^{n}(x_i - a_i)\frac{\partial f}{\partial x_i}(\mathbf{a})$.
For $k > 2$ in n variables: To approximate f at a given $\mathbf{x}$ near $\mathbf{a}$, let $g(t) = f(t\mathbf{x} + (1-t)\mathbf{a})$, so that $f(\mathbf{x}) = g(1)$ and $f(\mathbf{a}) = g(0)$; then, use the single-variable approximation around $t = 0$.

H. Linear algebra and linear equation systems. This note denotes matrices by boldface capitals or denotes them by their *elements:* a matrix $\mathbf{A} = \left(a_{ij}\right)_{i,j}$ of m rows and n columns has *order* $m \times n$. Minuscle boldface $\mathbf{v}$ indicates order $m \times 1$, a *column vector*. Order $1 \times n$ means a *row vector* and is denoted by $\mathbf{u}'$ where $\mathbf{u}$ is $n \times 1$ and the prime symbol $'$ denotes *matrix transpose:* if $\mathbf{A} = \left(a_{ij}\right)_{i,j}$ is $m \times n$, then $\mathbf{A}' = \mathbf{B}$ is the $n \times m$ matrix with $b_{ij} = a_{ji}$.
We write $\mathbf{0} = \mathbf{0}_{m,n}$ for a matrix with all elements being zero and $\mathbf{I} = \mathbf{I}_n$ for the (*square*) $n \times n$ matrix with elements $= 1$ on the *main diagonal* (i.e. if $i = j$) and 0 elsewhere.
If $\mathbf{A}$ is 1x1 we typically don't distinguish between the matrix $\mathbf{A}$ and the number a_{11}.

Scaling and addition. A matrix (and hence a vector) can be scaled by a number t, by scaling each element with t. We write $-\mathbf{A}$ for $(-1)\mathbf{A}$. Two matrices of the same order (hence also two vectors of the same order) are added element-wise.

Rules for scalings and sums. Scalings and sums of $m \times n$ matrices obey the rules $\mathbf{A}+\mathbf{0} = \mathbf{A}$; $\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$; $(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$ (so we drop the parentheses); $\mathbf{A}+(-\mathbf{A}) = \mathbf{0}$; $t(\mathbf{A}+\mathbf{B}) = t\mathbf{A}+t\mathbf{B}$; $(s+t)\mathbf{A} = s\mathbf{A}+t\mathbf{A}$. Subtraction is defined as $\mathbf{A}-\mathbf{B} = \mathbf{A}+(-\mathbf{B})$.

Products. For n-vectors $\mathbf{u}$ and $\mathbf{v}$, the dot product $\mathbf{u} \cdot \mathbf{v}$ is defined as $u_1 v_1 + \cdots + u_n v_n$. Also we define $\mathbf{u}' \cdot \mathbf{v}' = \mathbf{u} \cdot \mathbf{v}$ for row vectors of same order.
The matrix product $\mathbf{AB}$ is defined iff $\mathbf{A}$ resp. $\mathbf{B}$ have orders $m \times n$ resp. $n \times p$, and is the $m \times p$ matrix $\mathbf{C} = (c_{ij})$ with $c_{ij} = \mathbf{r}_i \cdot \mathbf{b}_j$, where $\mathbf{r}_i'$ is the ith row of $\mathbf{A}$ and $\mathbf{b}_j$ is the jth column of $\mathbf{B}$. *«Matrix division» is not defined,* though a 1×1 might be considered as a number.

Rules: products and transposition. Provided the matrix orders admit the operations, we have $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$ (so we drop these parentheses); $\mathbf{I}_m\mathbf{A} = \mathbf{A}\mathbf{I}_n = \mathbf{A}$; $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC}$; $(\mathbf{A}+\mathbf{B})\mathbf{C} = \mathbf{AC}+\mathbf{BC}$; $(\mathbf{A}')' = \mathbf{A}$; $(\mathbf{A}+\mathbf{B})' = \mathbf{A}'+\mathbf{B}'$; $(t\mathbf{A})' = t\mathbf{A}'$; and, $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.

Linear equation systems, general facts. A linear equation system $\mathbf{AX} = \mathbf{B}$ has either no solution, unique (= precisely one) solution, or infinitely many solutions.
If some solution $\mathbf{X}^*$ exists, the general solution – i.e. the set of all solutions – is of the form $\mathbf{X}^*$ plus the general solution of corresponding *homogeneous* equation system $\mathbf{AX} = \mathbf{0}$.
A homogeneous system $\mathbf{AX} = \mathbf{0}$ has at least one solution, namely the *trivial* solution $\mathbf{X} = \mathbf{0}$.

Gaussian elimination. On the augmented coefficient matrix $(\mathbf{A} \vdots \mathbf{B})$, delete on sight null rows (i.e. equations that say zero = zero), and apply the elementary row operations:

- Interchanging rows (i.e. equations);
- Scaling a row (i.e. an eq.) by a *nonzero* number (this to get leading 1's);
- Adding a scaling of one row (i.e. an eq.) to another (this to eliminate below leading 1's)

If and when an equation reads zero = something nonzero, you can declare «*no solution*».
Otherwise: If and when you have arrived at row-echelon form where each row has a leading 1 somewhere on the left-hand side, the corresponding variable numbers will be determined once the remaining $d \in \{0, 1, ...\}$ variables are chosen freely; «*solution with d degrees of freedom*».
Special case: $d = 0$ and unique solution. Then you can eliminate all the way to the left-hand side being $\mathbf{I}$. That is, an equation system of the form $\mathbf{IX} = \mathbf{M}$, with unique solution $\mathbf{X} = \mathbf{M}$.

Determinants and rules for determinants. If $\mathbf{A}$ is $n \times n$, we can define its *determinant*, a function denoted $\det(\mathbf{A})$ or $|\mathbf{A}|$. We say that $|\mathbf{A}|$ has order n (or even $n \times n$). The full definition is omitted (not needed!), but: $|\mathbf{A}|$ is the sum of $n!$ terms, each being $\pm$ the product of precisely one element from each row&column, the «$\pm$» chosen according to (H7) and $|\mathbf{I}_n| = 1$.

Let $\mathbf{A}$ and $\mathbf{B}$ both be $n \times n$. Then the following rules apply:

(H1) The *cofactor expansion rule* determines an order n determinant as a sum of n determinants each of order $n-1$: For $n = 1$, the determinant is the (only!) element of the matrix. For $n > 1$, let k_{ij} be the *cofactor* of element i, j, defined as $(-1)^{i+j}$ times the $(n-1) \times (n-1)$ determinant formed by deleting row i and column j from the matrix.

- Fix any row i; then $\quad |\mathbf{A}| = a_{i1}k_{i1} + \cdots + a_{in}k_{in}$

This is called *cofactor expansion along the ith row.* (Fact: *independent* of choice of i.)

(H2) $|\mathbf{A}'| = |\mathbf{A}|$. Hence cofactor expansion can be performed by arbitrary *column* as well: $|\mathbf{A}| = a_{1j}k_{1j} + \cdots + a_{nj}k_{nj}$ (cofactor expansion *along jth column*), any $j = 1, \ldots, n$.

(H3) $|\mathbf{AB}| = |\mathbf{A}| \cdot |\mathbf{B}|$.

(H4) If $\mathbf{A}$ has a row (/a column) of zeroes, or two proportional rows (/columns), then $|\mathbf{A}| = 0$.

(H5) If $\mathbf{B}$ is formed from $\mathbf{A}$ by scaling one single row (/column) by t, then $|\mathbf{B}| = t|\mathbf{A}|$.
In particular, $|t\mathbf{A}| = t^n|\mathbf{A}|$ (scaling all n rows by t).

(H6) If $\mathbf{B}$ is formed from $\mathbf{A}$ by adding to row $\#i$ a scaling of another row $\#\ell \neq i$ (/to column $\#j$ a scaling of another column $\#\ell \neq j$), then $|\mathbf{B}| = |\mathbf{A}|$.

(H7) If $\mathbf{B}$ is formed from $\mathbf{A}$ by interchanging two rows (/two columns), then $|\mathbf{B}| = -|\mathbf{A}|$.

Inverses and rules for inverses. Cramér's rule. A matrix $\mathbf{M}$ is called the *inverse of* $\mathbf{A}$ and denoted $\mathbf{A}^{-1}$, if $\mathbf{AM} = \mathbf{MA} = \mathbf{I}$. Then we call $\mathbf{A}$ *invertible.* It *must necessarily be square.*

The following rules apply if $\mathbf{A}$ is $n \times n$ (otherwise it cannot be invertible) and $\mathbf{B}$ has n rows:

(H8) If $\mathbf{AM} = \mathbf{I}_n$ or $\mathbf{MA} = \mathbf{I}_n$ then $\mathbf{A}$ is invertible with $\mathbf{A}^{-1}$ *uniquely* given by $\mathbf{M}$.
If so, then (since $(\mathbf{AM})' = \mathbf{M}'\mathbf{A}'$ also is $= \mathbf{I}_n$): $\mathbf{A}'$ will be invertible with inverse $\mathbf{M}'$.

(H9) If $\mathbf{A}$ is invertible, then $\mathbf{M} = \mathbf{A}^{-1}$ is invertible, and with inverse $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$. Also, for any natural number k: $\mathbf{A}^k$ will be invertible with inverse $(\mathbf{A}^{-1})^k$ (this denoted $\mathbf{A}^{-k}$).

(H10) $\mathbf{A}$ is invertible if and only if $|\mathbf{A}| \neq 0$. If so, then (by (H3)) $|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$.

(H11) $\mathbf{AB}$ is invertible if and only if $\mathbf{A}$ and $\mathbf{B}$ are both invertible. If so, $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.
If furthermore $t \neq 0$ then $t\mathbf{A} = \mathbf{A}(t\mathbf{I})$ is invertible with inverse $(t^{-1}\mathbf{I}^{-1})\mathbf{A}^{-1} = t^{-1}\mathbf{A}^{-1}$.

(H12) *Formula:* Let $\mathbf{K} = (k_{ij})$ be the matrix of cofactors of $\mathbf{A}$ (i.e.: each k_{ij} as defined in (H1)).
Then $\mathbf{AK}' = |\mathbf{A}|\,\mathbf{I}$. So (by (H8) and (H10)): if $\mathbf{A}$ is invertible, then $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\,\mathbf{K}'$.

(H13) If and only if $\mathbf{A}$ is invertible, then the equation system $\mathbf{AX} = \mathbf{B}$ has a *unique* solution (of same order $n \times p$ as $\mathbf{B}$, since $\mathbf{A}$ is square), and given by $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$. In particular: $\mathbf{AX} = \mathbf{I}$ has unique solution $\mathbf{X} = \mathbf{A}^{-1}$ (by (H8)) iff $\mathbf{A}$ invertible, no solution if not.

(H14) *Cramér's rule:* If and only if $\mathbf{A}$ is invertible, the unique solution of $\mathbf{Ax} = \mathbf{b}$ is given by $x_i = D_i/|\mathbf{A}|$ where D_i is the determinant formed by replacing column $\#i$ of $\mathbf{A}$ by $\mathbf{b}$.

I. Miscellaneous topics

The quadratic equation Provided $a \neq 0$, the equation $ax^2 + bx + c = 0$ has the solutions

(I1) $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{though no real solution if } b^2 < 4ac$$

Homogeneous functions. A function f of n variables $\mathbf{x} = (x_1, \ldots, x_n)$ is called *homogeneous of degree* d if for all $t > 0$ and all $\mathbf{x}$ in the domain of f, we have:

(I2) $$f(t\mathbf{x}) = f(tx_1, \ldots, tx_n) \text{ is defined and equals } t^d f(\mathbf{x}).$$

In particular, its domain D must be so that $\mathbf{x} \in D \Leftrightarrow t\mathbf{x} \in D$ for all $t > 0$. For such a domain and a C^1 function, the following are equivalent:

(I3) $$f \text{ homogeneous of degree } d \iff x_1 \frac{\partial f}{\partial x_i}(\mathbf{x}) + \cdots + x_n \frac{\partial f}{\partial x_n}(\mathbf{x}) = d \cdot f(\mathbf{x}) \quad \text{on } D$$

which provided $f(\mathbf{x}) \neq 0$, is equivalent to $\mathrm{El}_1 f(\mathbf{x}) + \cdots + \mathrm{El}_n f(\mathbf{x}) = d$ on D.
If f is C^1 and homogeneous of degree d, then each $\frac{\partial f}{\partial x_i}$ is homogeneous of order $d - 1$.
If furthermore f is C^2, then $\frac{\partial^2 f}{\partial x_i \, \partial x_j}$ homogeneous of order $d - 2$, every i, j, and

(I4) $$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j \frac{\partial^2 f}{\partial x_i \, \partial x_j}(\mathbf{x}) = d \cdot (d-1) \cdot f(\mathbf{x})$$

Homothetic functions. Let $D \subseteq R^n$ such that $\mathbf{x} \in D \Leftrightarrow t\mathbf{x} \in D$ for all $t > 0$. A function f defined on D is *homothetic* if

(I5) $$\text{whenever } f(\mathbf{u}) = f(\mathbf{v}), \text{ then } f(t\mathbf{u}) = f(t\mathbf{v}) \quad \text{for all } t > 0.$$

Any homogeneous function is homothetic. If h is homothetic and g is a strictly increasing function of a single variable, then $f(\mathbf{x}) = g(h(\mathbf{x}))$ is also homothetic.

The elasticity of substitution. Fix a level curve $F(K, L) = C$ of a function F of two variables. The *elasticity of substitution* $\sigma_{L,K}$ between K and L, measures the relative change in L/K per relative change in the *marginal rate of substitution* $R_{L,K} = \frac{F'_K(K,L)}{F'_L(K,L)}$ along the level curve:

(I6) $$\sigma_{L,K} = \mathrm{El}_{R_{L,K}} \frac{L}{K} = \frac{d \, \ln \frac{L}{K}}{d \, \ln \frac{F'_K(K,L)}{F'_L(K,L)}} \qquad \text{where } (K, L) \text{ such that } F(K, L) = C.$$

The elasticity of substitution can also be written as:

(I7) $$\sigma_{L,K} = \frac{F'_K F'_L}{KL} \cdot \frac{KF'_K + LF'_L}{B} \quad \text{where} \quad B = -F''_{KK}(F'_L)^2 + 2F'_K F'_L F''_{KL} - F''_{LL}(F'_K)^2$$

The latter denominator B equals $\begin{vmatrix} 0 & F'_K & F'_L \\ F'_K & F''_{KK} & F''_{KL} \\ F'_L & F''_{KL} & F''_{LL} \end{vmatrix}$ (the «*bordered Hessian*» determinant).