ECON3120/4120 Mathematics 2

postponed exam, January 19th 2024 (4 hrs, English only)
There are 4 problems to be solved.
Support material: "Rules and formulas" attachment.

For the entire problem set:

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier enumerated item (e.g. "(a)") to solve a later one (e.g. "(c)"), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Problem 1 For each real constant t define the matrices $\mathbf{A}_t$ and $\mathbf{B}_t$ and the vector $\mathbf{v}_t$ as

$$\mathbf{A}_t = \begin{pmatrix} 0 & 0 & 0 & t \\ 0 & 0 & t^2 & 0 \\ 0 & t^3 & 3 & 0 \\ t^4 & 0 & 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B}_t = \begin{pmatrix} t & 4 & 0 & -t^2 \\ 0 & 3t & -t^3 & 0 \\ 0 & -t^4 & 0 & 0 \\ -t^5 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_t = \begin{pmatrix} 0 \\ t^2 \\ 3 \\ 4 \end{pmatrix}$$

Note that $\mathbf{v}_t$ is the third column of $\mathbf{A}_t$.

(a) Calculate $\mathbf{v}_t'\mathbf{v}_t$, $\mathbf{A}_t\mathbf{v}_t$ and $\mathbf{A}_t\mathbf{B}_t$. (The latter should get you a *diagonal* matrix!)

(b) Calculate the determinants of $\mathbf{A}_t$ and of the matrix $\mathbf{M}_t = 2t\mathbf{A}_t$.

(c) Show that the equation system $\mathbf{A}_t\mathbf{x} = \mathbf{v}_t$ always has a solution, no matter what t is.

(d) For those t that $\mathbf{A}$ has an inverse: Use part (a) to find an expression for $\mathbf{A}_t^{-1}$.

Problem 2 Let $R > 0$ be constant. For each $R > 0$, the following functions are defined for all positive x: $g(x) = \frac{1}{x} + \ln x - R - Rx$ and $h(x) = (\ln x - Rx)e^{-x}$.

It is a fact that $h'(x) = g(x)e^{-x}$.

(a) Show the following limits:

(i) $\lim_{x \to 0^+} x \ln x = 0$

(ii) $\lim_{x \to 0^+} g(x) = +\infty$ (*Hint:* $\frac{1}{x} + \ln x = \frac{1}{x} \cdot \left[1 + x \cdot \ln x\right]$)

(iii) $\lim_{x \to +\infty} g(x) = -\infty$

(b) Use part (a) to show that h has *at least one* stationary point p.
(You are not asked to find p. *Hint:* The fact that $h'(x) = g(x)e^{-x}$ means h increases where g is positive, decreases when h is negative, and is stationary when ... ?)

(c) p depends on R. Find an expression for $\dfrac{dp}{dR}$.

(d) Take for granted that $x = p$ is a global maximum for h. Then the maximum *value* $V = h(p)$ depends on R. Find an expression for $\dfrac{dV}{dR}$.

Problem 3

(a) *Use the substitution* $u = -\ln z$ to show that

$$\int \ln z \, dz = z \cdot (\ln z - 1) + C$$

If you are unable to do so using that substitution, you can get up to the pass mark for showing the formula by any method of your choice.

(b) Find the general solution of the differential equation

$$\dot{x} = x^2 \cdot \ln t$$

Problem 4 Let $f(x,y) = \ln x + 2\ln y$, and consider the problems

$$\max / \min\ f(x,y) \quad \text{subject to} \quad x^2 + y^2 = 3 \qquad \text{(L)}$$
$$\max\ f(x,y) \quad \text{subject to} \quad x^2 + y^2 \leq 3 \qquad \text{(K)}$$

Note that there are two "(L)" problems, one max and one min, but only a maximization problem under inequality constraint.

(a) Consider the Lagrange conditions associated to problems (L).
Show that there is precisely one point $(\tilde{x}, \tilde{y})$ that satisfies these conditions, and *find* this point.
(*Hint:* Eliminate $1/\lambda$ to find $\frac{1}{\lambda} = 2x^2 = y^2$.)

(b) The constraint $x^2 + y^2 = 3$ forms a circle (with radius $\sqrt{3}$), and point $(\tilde{x}, \tilde{y})$ lies on that circle. The following argument is nevertheless flawed; *find the flaw in the argument:*

> « *The circle formed by the constraint is a closed and bounded and nonempty set. The function f is continuous. Hence the extreme value theorem grants the existence of a max and a min, and both will have to be at the only possible point* $(\tilde{x}, \tilde{y})$. »

(c) Does point $(\tilde{x}, \tilde{y})$ satisfy the Kuhn–Tucker conditions associated with problem (K)?

(d) One of the following is true, and you shall prove the true one:

- Prove that $(\tilde{x}, \tilde{y})$ solves problem (K).

  OR:

- Prove that $(\tilde{x}, \tilde{y})$ solves the *minimization* (L).

(End of problem set. Attachment: Rules and formulas.)