University of Oslo / Department of Economics

ECON3120/4120 Mathematics 2: on the 2024-01-19 exam

- This is a postponed exam. The general principles of an ordinary exam applies – see a guideline on a (recent, 4h) ordinary Mathematics 2 exam.
- Ordinary exam was maybe too much work, which had to be compensated in grading. This set is less. In ordinary grading: some papers (obviously only a few, given the low total number) did benefit from «generous round-offs» compared to the usual grading thresholds. In case of appeals: up to the appeals committee's judgement.

Next pages: Problems (restated as given) and solutions and annotations (boxed) follow. Page number = problem number.

Problem 1 For each real constant t define the matrices $\mathbf{A}_t$ and $\mathbf{B}_t$ and the vector $\mathbf{v}_t$ as

$$\mathbf{A}_t = \begin{pmatrix} 0 & 0 & 0 & t \\ 0 & 0 & t^2 & 0 \\ 0 & t^3 & 3 & 0 \\ t^4 & 0 & 4 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B}_t = \begin{pmatrix} t & 4 & 0 & -t^2 \\ 0 & 3t & -t^3 & 0 \\ 0 & -t^4 & 0 & 0 \\ -t^5 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_t = \begin{pmatrix} 0 \\ t^2 \\ 3 \\ 4 \end{pmatrix}$$

Note that $\mathbf{v}_t$ is the third column of $\mathbf{A}_t$.

(a) Calculate $\mathbf{v}_t'\mathbf{v}_t$, $\mathbf{A}_t\mathbf{v}_t$ and $\mathbf{A}_t\mathbf{B}_t$. (The latter should get you a *diagonal* matrix!)

(b) Calculate the determinants of $\mathbf{A}_t$ and of the matrix $\mathbf{M}_t = 2t\mathbf{A}_t$.

(c) Show that the equation system $\mathbf{A}_t\mathbf{x} = \mathbf{v}_t$ always has a solution, no matter what t is.

(d) For those t such that $\mathbf{A}$ has an inverse: Use part (a) to find an expression for $\mathbf{A}_t^{-1}$.

Notes:

- The following omits the subscript. It is in the problem to emphasize that there is a dependence upon t.
- For (a): This class has seen «many matrix products between vectors». Not to make it completely predictable, some would have to be column times row and and some have to be row times column – which is 1×1 in which case *matrix delimiters* are pretty much optional: it is common not to distinguish between such a matrix and its element.
- For (c): From the fact that $\mathbf{v}$ is the third column of $\mathbf{A}$, this $(0,\ 0,\ ,1,\ 0)'$ is a solution no matter t. It isn't expected to spot, and the presence of that «hint» was due to an early draft which had a question where it could be more useful.

How to solve:

(a) $\mathbf{v}'\mathbf{v} = 0 \cdot 0 + t^2 \cdot t^2 + 3 \cdot 3 + 4 \cdot 4 = t^4 + 25$

$$\mathbf{A}\mathbf{v} = \begin{pmatrix} 0 + t \cdot 4 \\ 0 + t^2 \cdot 3 + 0 \\ 0 + t^3 \cdot t^2 + 3 \cdot 3 + 0 \\ 0 + 0 + 4 \cdot 3 + 1 \cdot 4 \end{pmatrix} = \begin{pmatrix} 4t \\ 3t^2 \\ t^5 + 9 \\ 16 \end{pmatrix}$$

$$\mathbf{A}\mathbf{B} = \begin{pmatrix} 0 + t \cdot (-t^5) & 0 & 0 & 0 \\ 0 & 0 + t^2 \cdot (-t^4) & 0 & 0 \\ 0 & t^3 \cdot 3t - 3t^4 & -t^6 & 0 \\ t^4 \cdot t - t^5 & 4t^4 - 4t^4 & 0 & 0 \end{pmatrix} = -t^6\mathbf{I}$$

(b) Cofactor expansion along first column and then by first column again, yields $|\mathbf{A}| = -t^4 \cdot$ <top-right 3×3 minor> $= -t^4 \cdot t^3 \cdot \begin{vmatrix} 0 & t \\ t^2 & 0 \end{vmatrix} = t^{10}$. Then – because $\mathbf{A}$ is 4×4 – we get $|\mathbf{M}| = (2t)^4|\mathbf{A}| = 16 \cdot t^{14}$.

(c) Unique solution for $t \neq 0$ so only the case $t = 0$ remains. But then the equation system reduces to $0 = 0$, $0 = 0$, $3x_3 = 3$, $4x_3 + x_4 = 4$ which has solutions, for example $(0,\ 0,\ 1,\ 0)'$.

(d) $\mathbf{A}^{-1} = t^{-6}\mathbf{B}$ since $\mathbf{A}$ is square and from (a) we have $\mathbf{A}(-t^{-6}\mathbf{B}) = -t^{-6}(-t^6\mathbf{I}) = \mathbf{I}$, all $t \neq 0$.

Problem 2 Let $R > 0$ be constant. For each $R > 0$, the following functions are defined for all positive x: $g(x) = \frac{1}{x} + \ln x - R - Rx$ and $h(x) = (\ln x - Rx)e^{-x}$.

It is a fact that $h'(x) = g(x)e^{-x}$.

(a) Show the following limits:

(i) $\lim_{x\to 0^+} x \ln x = 0$

(ii) $\lim_{x\to 0^+} g(x) = +\infty$ (*Hint:* $\frac{1}{x} + \ln x = \frac{1}{x} \cdot \big[1 + x \cdot \ln x\big]$)

(iii) $\lim_{x\to +\infty} g(x) = -\infty$

(b) Use part (a) to show that h has *at least one* stationary point p.
(You are not asked to find p. *Hint:* The fact that $h'(x) = g(x)e^{-x}$ means h increases where g is positive, decreases when h is negative, and is stationary when ... ?)

(c) p depends on R. Find an expression for $\dfrac{dp}{dR}$.

(d) Take for granted that $x = p$ is a global maximum for h. Then the maximum *value* $V = h(p)$ depends on R. Find an expression for $\dfrac{dV}{dR}$.

Note: Both $\mathbf{b}$ and the *transpose of* $\mathbf{M}$ appeared in both compulsory hand-ins 3 and 4; glyphs «w» and «$\mathbf{M}$» did differ. Hand-in 3 had a multiplication exercise and hand-in 4 had determinant and an inverse. Maybe some will recognize the determinant after having calculated it.

How to solve:

(a) (i) $\lim_{x\to 0^+} x \ln x = \overbrace{\lim_{x\to 0^+} \frac{\ln x}{x^{-1}}}^{«-\infty/\infty»} = \lim_{x\to 0^+} \frac{x^{-1}}{-x^{-2}} = \lim_{x\to 0^+}(-x) = 0.$

(ii) From the hint, $\frac{1}{x} + \ln x = \frac{1}{x}\big[1 + x \ln x\big]$ where from (i) the bracketed term $\to 1 + 0$. So $\frac{1}{x} + \ln x \to «1/0^+» = +\infty$, and $g(x) \to +\infty - R - 0 = +\infty$.

(iii) When $x \to +\infty$, $g(x) \to 0 - R + \lim(\ln x - Rx)$. There are many ways to show that the latter limit is $-\infty$, and for exam purposes one is of course enough. Possible ways, the first analogous to (ii) and the next should be known from compulsory hand-ins:

- $\ln x - Rx = x \cdot \big(\frac{\ln x}{x} - R\big)$ and since $\frac{\ln x}{x} \to \infty/\infty$ here as well, that limit is $= \lim 1/x = 0$ and we get $\lim x \cdot (0 - R) = -\infty$.
- $\ln x - Rx$ is concave, has stationary point (hence global max) where $1/x = R$, and will from then on decrease steeper and steeper. Since we go to $+\infty$, it will $\to -\infty$.
- $(\ln x - Rx)' = \frac{1}{x} - R \to -R$, so from some point $\hat{x}$ on, the derivative is $< -R/2$ (say). For larger x then $\ln x - Rx = \ln \hat{x} - R\hat{x} + \int_{\hat{x}}^{x} (\frac{1}{t} - R)dt < \ln \hat{x} - R\hat{x} - (x - \hat{x})R/2$ which $\to -\infty$.

(b) h has stationary point when $g = 0$. We have that $g(x_*) > 0$ for some small x_* (indeed all small enough) and $g(x^*) < 0$ for some large x^* (indeed all x^* large enough), so by the intermediate value theorem, g has a zero p between x_* and x^*.

(c) Upgrade R to a variable, let $\gamma(p, R) = \frac{1}{p} + \ln p - R - Rp$. Then $\gamma = 0$, and $\dfrac{dp}{dR} = -\dfrac{\gamma'_p(p, R)}{\gamma'_R(p, R)} = \dfrac{-p^{-2} + p^{-1} - R}{1 + p}$. (You can prettify if you like.)

(d) By the envelope theorem, $V'(R) = \dfrac{\partial}{\partial R}\Big[(\ln x - Rx)e^{-x}\Big]\Big|_{x=p} = -pe^{-p}$.

Problem 3

(a) *Use the substitution* $u = -\ln z$ to show that

$$\int \ln z\, dz = z \cdot (\ln z - 1) + C$$

If you are unable to do so using that substitution, you can get up to «E» worth of score for showing the formula by any method of your choice.

(b) Find the general solution of the differential equation

$$\dot{x} = x^2 \cdot \ln t$$

How to solve:

(a) The substitution yields $du = -z^{-1}dz$, where $z = e^{-u}$ and $dz = -e^{-u}\, du$. The integral transforms to $\int(-u)(-e^{-u})du$. By parts, with $v' = e^{-u}$ and $v = -e^{-u}$, we have $uv - \int \frac{du}{du}\big(-e^{-u}\big)du = -ue^{-u} - \int(-e^{-u})du = -ue^{-u} - e^{-u} + C = z\ln z - z + C$ as should.

(The alternative method for «E»: Differentiate right-hand side.)

(b) Separable differential equation with constant solution for $x = 0$. For nonzero x, separate and integrate:

$$\int x^{-2}dx = \int (\ln t)dt \qquad \Longrightarrow \qquad \frac{-1}{x} = t(\ln t - 1) + C \quad \text{(from (a))}$$

and so the solution is $x = \dfrac{-1}{C + t(\ln t - 1)}$ or $x = 0$.

Note: You cannot obtain $x = 0$ as a special case of $x = \dfrac{-1}{C + t(\ln t - 1)}$, with any real constant C. That was deliberate.

Problem 4 Let $f(x,y) = \ln x + 2\ln y$, and consider the problems

$$\max / \min\ f(x,y) \quad \text{subject to} \quad x^2 + y^2 = 3 \tag{L}$$
$$\max\ f(x,y) \quad \text{subject to} \quad x^2 + y^2 \leq 3 \tag{K}$$

Note that there are two "(L)" problems, one max and one min, but only a maximization problem under inequality constraint.

(a) Consider the Lagrange conditions associated to problems (L).
Show that there is precisely one point $(\tilde{x}, \tilde{y})$ that satisfies these conditions, and *find* this point.
(*Hint:* Eliminate $1/\lambda$ to find $\frac{1}{\lambda} = 2x^2 = y^2$.)

(b) The constraint $x^2 + y^2 = 3$ forms a circle (with radius $\sqrt{3}$), and point $(\tilde{x}, \tilde{y})$ lies on that circle. The following argument is nevertheless flawed; *find the flaw in the argument:*

> *« The circle formed by the constraint is a closed and bounded and nonempty set. The function f is continuous. Hence the extreme value theorem grants the existence of a max and a min, and both will have to be at the only possible point $(\tilde{x}, \tilde{y})$. »*

(c) Does point $(\tilde{x}, \tilde{y})$ satisfy the Kuhn–Tucker conditions associated with problem (K)?

(d) One of the following is true, and you shall prove the true one:
- Prove that $(\tilde{x}, \tilde{y})$ solves problem (K).
  OR:
- Prove that $(\tilde{x}, \tilde{y})$ solves the *minimization* (L).

How to solve: We will need the Lagrangian $f(x,y) - \lambda(x^2 + y^2 - 3)$ and its partial derivatives $L'_x(x,y) = \frac{1}{x} - 2\lambda x$ and $L'_y(x,y) = \frac{2}{y} - 2\lambda y$.

(a) Conditions are then $\frac{1}{x} = 2\lambda x$, $\frac{2}{y} = 2\lambda y$ and $x^2 + y^2 = 3$. As per the hint, $\frac{1}{\lambda} = 2x^2$ and $\frac{2}{\lambda} = 2y^2$ and hence $2y^2 = \frac{2}{\lambda} = 2 \cdot 2x^2$. Inserting $2x^2$ for y^2 we get $3 = x^2 + 2x^2$, so $x = 1$. (Only the positive square root, as ln is not defined at -1.) Then $y^2 = 2$ so $y = \sqrt{2}$ (again only the positive, due to the ln).
Conclusion: Lagrange point $\Leftrightarrow (x,y) = (\tilde{x}, \tilde{y}) = (1, \sqrt{2})$.

(b) The flaw is: f isn't defined on the entire circle. Thus not continuous on the circle.
(f is a «continuous» function, but that means «continuous where it is defined».)

(c) Constraint is active, so Kuhn–Tucker holds if and only if $\lambda \geq 0$. That holds: $\lambda = (2x^2)^{-1} > 0$.

(d) Because the point satisfies the Kuhn–Tucker conditions and L is concave (being a sum of concave functions, because $\lambda \geq 0$), the point solves the Kuhn–Tucker maximization problem.

Note on (d) (cf. (c)): Point $(\tilde{x}, \tilde{y})$ does not solve the minimization part of (L), so certainly any attempt of proving so would be futile. Even without considering concavity, there should be ample hints that one should not even attempt, especially if one spots the point of part (c): as we move along the circle to the boundary of the domain of f, the log terms will make $f \to -\infty$, and so no minimum can exist. Or if one was able to locate $(\tilde{x}, \tilde{y})$ one could evaluate $f(1, \sqrt{2}) = \ln 1 + 2\ln 2^{1/2} = \ln 2$ and then pick some other point on the circle (arbitrarily, though in the first orthant so that f is defined); say, $f(\sqrt{2}, 1) = \frac{1}{2}\ln 2 + 2\ln 1 = \frac{1}{2}\ln 2$ which is $< \ln 2$, disproving minimum immediately.