University of Oslo / Department of Economics

ECON3120/4120 Mathematics 2: on the 2023-12-18 exam

- *Standard disclaimer:* A note like this is not suited as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
  - The document reflects what was expected in that particular semester, and which may not be applicable to future semesters. In particular, what tests one is required to perform before answering «no conclusion» may not apply for later.[*]
- Weighting: Problem set *suggested* uniform weighting over letter-enumerated items (14 in total), leaving a decision up to the committee (and in case of appeals: the new grading committee).
- The 2019 exam format change was not intended to change overall requirements. Rather, it was intended to facilitate better differentiation between candidates.
  - Grading scale had been defaulting to thresholds 91–75–55–45–40 more or less since the ECTS letter grades were introduced. Upon the change in exam format in 2019, a slightly easier set was intended to facilitate the 92–77–58–46–40 scale once recommended by the Norwegian Mathematical Council and commonly applied at The Faculty of Mathematics and Natural Sciences.
  - It was a hope that the effort put into the 2019 grading would set a practice[†] – but as it turned out, the next exam had the COVID format. It would be a stretch to say that one *established* any change in grading scale.
- 2019 also saw a major change in compulsory activities (with slight revision in 2022). More problems were made compulsory than in the age of 3h exams, and one could no longer bring the problems and solutions to the exam; thus making it possible to align parts of the exam more closely up to parts of the compulsory problem sets.
  - This document indicates and comments upon those parts that more closely follow compulsory hand-ins. Whether those are then, ceteris paribus, to be considered «easier» since they should be more spot-on familiar, is subject to the committee's discretion.
  - The compulsory problem sets are also available to the committee upon request.

[*] The standard disclaimer was kept even if the 2023-12-18 set has no such classification problem that could lead to a «no conclusion» answer.

[†] Quoting from the 2019 grading guideline:
« Given that this is the first exam set in the new format, it might set standards for the years to come, and the committee should set benchmarks with caution. There might be less reason to stress the percentage-to-grade tables that have been applied earlier (nominally defaulting to 91–75–55–45–40); the 40 percent pass mark does however hold a long history and as a preliminary view I would consider it to be more of a constant than the other thresholds. »
The 2019 exam(s) did end up applying 92–77–58–46–40.

Addendum after grading: An appeals committee is free to act upon their own discretion. In the course of ordinary grading, the following principles ended up being applied, after some weighing back and forth:

- For *most* candidates, problems 2(b) and 4(d) were effectively disregarded. Then uniform weighting was applied over the remaining twelve letter-enumerated items.
  - Hardly anyone got 2(b). One could have then distinguished out those who knew the concept from those who didn't seem to understand the question. Zero-weighting was considered less arbitrary, and one could argue the problem was too hard.
  - Similar considerations could apply to 4(d), although under considerable doubt. We do believe that candidates should be expected to recognize the significance of concavity in maximization with or without constraints. On the other hand, the problem set might have had more than enough questions in total, and that often hurts the score on the last question in the set.
- Nevertheless, positive score on 2(b) and/or 4(d) were taken into account for candidates sufficiently near a grading threshold. Some candidate(s) did end up with a better grade than they would have under zero-weighting.
- Without those two questions, the problem set appears to be on the easy side. It is not clear-cut, as other problem sets have assigned a full question on writing out Kuhn–Tucker conditions. Nevertheless, it was often hard to distinguish the best from the very good. The committee did not get out their microscopes trying to find details that could reduce the number of «A» grades – it would, in the committee's opinion, not make the assessment more fair to on the (very good or even better) papers.

A fix-up (for use in teaching later): 2(b), where the original version was maybe not clear enough on how to find $|\mathbf{M}_w|$ and how to write it out. Also some linebreaks were prettified for readability.

Next pages: Problems (restated as given) and solutions and annotations (boxed) follow:

Problem 1 Take for granted that the equation system

$$\begin{aligned} (4-x)^2 + yx^{3/2} - t &= 5 \\ 16x + 3ye^{x-1} &= 24 \end{aligned} \tag{S}$$

determines continuously differentiable functions $x = x(t)$ and $y = y(t)$ around the point where $(t, x, y) = (\frac{20}{3}, 1, \frac{8}{3})$.

(a) Differentiate the equation system.

(b) Use the differentiated system to find an approximate value for $x(7)$.

Note: Although it is possible to eliminate y from the system as is, the phrasing «Use the differentiated system to» in part (b) is intended to make it clear they are expected to use precisely that.

How to solve:

(a) Calculating differentials:

$$\begin{aligned} -2(4-x)\,dx + \tfrac{3}{2}x^{1/2}y\,dx + x^{3/2}\,dy - dt &= 0 \\ 16\,dx + 3ye^{x-1}\,dx + 3e^{x-1}\,dy &= 0 \end{aligned}$$

(b) Either write $x(7) \approx x({}^{20}\!/\!_3) + x({}^{20}\!/\!_3)(7 - {}^{20}\!/\!_3)$ or simply put $dt = 7 - {}^{20}\!/\!_3 = {}^{1}\!/\!_3$ and write $x(7) \approx x({}^{20}\!/\!_3) + dx = 1 + dx$. We need dx. Inserting for the point, the differentiated system becomes

$$\begin{aligned} \big[-2\cdot 3 + \tfrac{3}{2}\cdot 83\big]\,dx + dy &= \tfrac{1}{3} \\ (16+8)\,dx + 3\,dy &= 0 \qquad \text{so } dy = -8\,dx \end{aligned}$$

and then $(-2)dx - 8dx = {}^{1}\!/\!_3$ and $\boxed{x(7) \approx 1 - {}^{1}\!/\!_{30} = \dfrac{29}{30}}$.

Problem 2 For each real constant w, define $\mathbf{M}_w = \begin{pmatrix} 2 & -5 & 0 \\ 1 & 0 & -3 \\ 0 & 4 & w \end{pmatrix}$ and $\mathbf{b}_w = \begin{pmatrix} 0 \\ 4 \\ w \end{pmatrix}$. Let $\mathbf{S} = \mathbf{M}'_w\mathbf{M}_w$, $\mathbf{T} = \mathbf{b}_w\mathbf{b}'_w$, $\mathbf{U} = (\mathbf{b}_w)^2$ and $\mathbf{V} = \mathbf{M}_w\mathbf{b}_w$, *provided they are well-defined.*

(a) For each of the matrix products $\mathbf{S}$, $\mathbf{T}$, $\mathbf{U}$, $\mathbf{V}$:

- Calculate both the last row and the last column – or explain why the product does not exist. (Answers might depend on w.)
- When that last row is the transpose of the last column (and for at least one product it is): Could we tell that before multiplying out – or is that "a mere coincidence" by the particular elements of $\mathbf{M}_w$ and/or $\mathbf{b}_w$?

(b) For each of the matrices $\mathbf{M}_w$ and $\mathbf{S}$ and $\mathbf{V}$:
Calculate the determinant or point out why the determinant is not well-defined.

(c) For what value(s) of w will the equation system $\mathbf{M}_w\mathbf{x} = \mathbf{b}_w$ have:
No solution? Precisely one solution? Several solutions?

(d) Let $w = 0$. *Use Cramér's rule* to show that $x_2 = 0$. (There is no score for any other method.)

Note: Both $\mathbf{b}$ and the *transpose of* $\mathbf{M}$ appeared in both compulsory hand-ins 3 and 4; glyphs «w» and «$\mathbf{M}$» did differ. Hand-in 3 had a multiplication exercise and hand-in 4 had determinant and an inverse. Maybe some will recognize the determinant after having calculated it.

How to solve:

(a) The last bullet item first: A product $\mathbf{AA}'$ is always symmetric. That goes for $\mathbf{S}$ (with $\mathbf{A} = \mathbf{M}'_w$) and $\mathbf{T}$ (with $\mathbf{A} = \mathbf{b}_w$). On to the first bullet item:

$\mathbf{S}$: Its last column is $\mathbf{M}' \begin{pmatrix} 0 \\ -3 \\ w \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ -5 & 0 & 4 \\ 0 & -3 & w \end{pmatrix} \begin{pmatrix} 0 \\ -3 \\ w \end{pmatrix} = \begin{pmatrix} 0-3+0 \\ 0+0+4w \\ 0+9+w^2 \end{pmatrix} = \boxed{\begin{pmatrix} -3 \\ 4w \\ 9+w^2 \end{pmatrix}}$, and by symmetry the last row is $\boxed{(-3,\ 4w,\ 9+w^2).}$

$\mathbf{T}$: Its last column is the 3×1 by 1×1 matrix product $\mathbf{b}_w\,(w)$ (the latter a matrix of order 1×1) and we get $w\mathbf{b}_w = \boxed{\begin{pmatrix} 0 \\ 4w \\ w^2 \end{pmatrix}}$. By symmetry the last row of $\mathbf{T}$ is $\boxed{(0,\ 4w,\ w^2).}$

$\mathbf{U}$: $\boxed{\text{Does not exist,}}$ only square matrices can be squared.

$\mathbf{V}$ equals $\begin{pmatrix} 0-20+0 \\ 0+0-3w \\ 16+w^2 \end{pmatrix} = \begin{pmatrix} -20 \\ -3w \\ 16+w^2 \end{pmatrix}$ and $\boxed{\text{that is also its last column.}}$ Its last row is the 1×1 matrix $\boxed{(16+w^2).}$

(b) By cofactor expansion along the first row:
$|\mathbf{M}_w| = 2\begin{vmatrix} 0 & -3 \\ 4 & w \end{vmatrix} - (-5)\begin{vmatrix} 1 & -3 \\ 0 & w \end{vmatrix} = 2 \cdot (0+12) + 5 \cdot (w-0) = \boxed{24+5w.}$
$|\mathbf{S}| = |\mathbf{M}'_w||\mathbf{M}_w| = |\mathbf{M}_w|^2 = \boxed{(24+5w)^2.}$
$\mathbf{V}$ is not square and therefore $\boxed{\text{has no determinant.}}$

(c) We have unique solution if and only if $|\mathbf{M}_w| \neq 0$, i.e. $\boxed{\text{if and only if } w \neq -{}^{24}/_{5}.}$
Only the case $w = -{}^{24}/_{5}$ remains:

$$\left(\begin{array}{ccc|c} 2 & -5 & 0 & 0 \\ 1 & 0 & -3 & 4 \\ 0 & 4 & w & w \end{array}\right) \begin{array}{l} {}^{-1/2} \\ {}^{+} \\ \end{array} \sim \left(\begin{array}{ccc|c} 2 & -5 & 0 & 0 \\ 0 & {}^{5}/_{2} & -3 & 4 \\ 0 & 4 & w & w \end{array}\right) \begin{array}{l} \\ {}^{-8/5} \\ {}^{+} \end{array}$$

and in the end the last row will say $(w + {}^{24}/_{5})x_3 = w - {}^{32}/_{5}$. When $w = -{}^{24}/_{5}$, it says $0 =$ <something nonzero> and $\boxed{\text{no solution exists in this case.}}$

(d) Replace the second column of $\mathbf{M}_w$ by $\mathbf{b}_w$, and the last row becomes $(0,\ w,\ w)$ which vanishes when $w = 0$. Thus $\boxed{x_2 = \frac{0}{|\mathbf{M}_0|} = 0.}$

Problem 3

(a) Show that

$$\int \frac{e}{(e-x)x}\,dx = C - \ln\left|\frac{1}{e} - \frac{1}{x}\right|$$

(b) Does the integral $\displaystyle\int_{e/2}^{2e} \frac{e}{(e-x)x}\,dx$ converge?

(c) *Use integration by parts* to find constants A and B such that

$$\int_1^t 2s\ln(s^3e^4)\,ds = A\cdot(t^2-1) + B\cdot t^2\ln t$$

(d) Consider the differential equation

$$\dot{x} = (e-x)x \cdot t\ln(t^3e^4)$$

Find the following *two* particular solutions: The one such that $x(1) = e$, and the one such that $x(1) = e/2$.

You are free to express your answer(s) in terms of the symbols «A» and «B» without inserting the actual numbers from part (c).

Notes:

(a) Replace «e» by 1 and flip sign inside the absolute value, and part (a) becomes precisely like in hand-in 4. Which was covered in an extra plenary seminar one week before, and they were urged to differentiate the right-hand side. Partial fractions is not syllabus, but those who use it cannot be penalized upon grading.

(b) is likely to see many wrong answers, especially since the pole is inside the scope of integration. But the wording, with «converge», should give sufficient hint that there is something to check.

(c) A question in hand-in 3 had (with different letters) $s\ln(es^2)$. While it was intended to rewrite $u = \ln(s^3e^4) = \ln e^4 + \ln s^3 = 4 + 3\ln s$, it isn't mandatory; in any case, one will obtain $du = \frac{3}{s}\,ds$ (with or without the intermediate $du = \frac{1}{s^3e^4}\cdot 3s^2e^4\,ds$.)

(d) It is intentional to ask for the constant solution, to catch those who divide without checking for zeroes. That is a grave sin in this course.

How to solve:

(a) $\Big(-\ln\big|\frac{1}{e}-\frac{1}{x}\big|\Big)' = -\dfrac{1}{{}^1\!/\!{}_e - {}^1\!/\!{}_x}\cdot\dfrac{1}{x^2} = \dfrac{-1}{{}^x\!/\!{}_e - 1}\cdot\dfrac{1}{x} = \dfrac{e}{(e-x)x}$, *OK!*

(b) The integrand is unbounded around $x = e$, which is inside the scope of integration. To converge, the improper integrals $\int_{e/2}^{e}$ and $\int_{e}^{2e}$ must both converge. Checking the former:

$$\lim_{q\to e^-}\int_{e/2}^{q}\frac{e}{(e-x)x}dx = \ln\Big|\frac{1}{e}-\frac{1}{e/2}\Big| - \underbrace{\lim_{q\to e^-}\ln\Big|\frac{1}{e}-\frac{1}{q}\Big|}_{\text{« }\ln 0^+\text{ »}}$$

$\boxed{\text{The integral does } \textit{not} \text{ converge.}}$

(c) With $u = 4+3\ln s$ and $v = s^2$, $dv = 2s\,ds$:

$$\int_1^t 2s\big(4+3\ln s\big)\,ds = \Big[s^2\cdot\big(4+3\ln s\big)\Big]_{s=1}^{s=t} - \int_1^t \overbrace{s^2\cdot\frac{3}{s}}^{=3s}ds = 3t^2\ln t + 4t^2 - 4 - \frac{3}{2}\Big[s^2\Big]_{s=1}^{s=t}$$
$$= (4-\tfrac{3}{2})(t^2-1) + 3t^2\ln t$$

so that $\boxed{A = 5/2 \text{ and } B = 3}$.

(d) $(e-x)x$ has zeroes for $x = 0$ or $x = e$, and so the first of the two particular solutions is the constant solution $\boxed{x(t)\equiv e.}$

The other solution hits value 2, and is not constant; furthermore, $2 > 0$ and $e-2 > 0$. We separate and integrate:

$$\int\frac{1}{(e-x)x}dx = \int t\ln(t^3e^4)\,dt \qquad \text{(Multiply LHS by } \tfrac{e}{e} \text{ and RHS by } \tfrac{2}{2}\text{)}$$
$$\tfrac{1}{e}\cdot\Big(-\ln\big|\tfrac{1}{e}-\tfrac{1}{x}\big|\Big) = K + \tfrac{1}{2}\Big[A(t^2-1) + Bt^2\ln t\Big] \quad \text{using (a) and (c)}$$

where K is the constant of integration to be determined from $x(1) = {}^e\!/\!{}_2$:

$$\tfrac{1}{e}\cdot\Big(-\ln\big|\tfrac{1}{e}-\tfrac{1}{e/2}\big|\Big) = K + 0$$

Observe that the phrase inside the absolute value sign is $\frac{1}{e}-\frac{2}{e} = -\frac{1}{e} < 0$, so for this particular solution: $\big|\frac{1}{e}-\frac{1}{x}\big| = \frac{1}{x}-\frac{1}{e}$. Furthermore, $K = \frac{1}{e}\cdot(-\ln(e^{-1}) = \frac{1}{e}$. Therefore,

$$\ln\big(\tfrac{1}{x}-\tfrac{1}{e}\big) = -1 - \tfrac{e}{2}\Big[A(t^2-1) + Bt^2\ln t\Big]$$
$$\frac{1}{x} = e^{-1} + \exp\big(-1 - \tfrac{e}{2}[A(t^2-1) + Bt^2\ln t]\big)$$
$$x = \boxed{\Big(e^{-1} + \exp\big(-1 - \tfrac{e}{2}[A(t^2-1) + Bt^2\ln t]\big)\Big)^{-1}}$$

(or, if you prefer, you can write as $\dfrac{e}{1+\exp\big(-\frac{e}{2}[A(t^2-1)+Bt^2\ln t]\big)}$, and/or rewrite $\exp(-\frac{e}{2}\cdot Bt^2\ln t) = \exp((\ln t)\cdot[-\frac{e}{2}\cdot Bt^2] = t^{-Bet^2/2}$, and/or insert for $A = {}^5\!/\!{}_2$ and $B = 3$.)

Problem 4 Let $F(x, y) = x + y - xy - x^2 - y^2$.

(a) Is F homogeneous? *Hint:* Euler's theorem.

Consider first the problems

$$\max F(x,y) \quad \text{subject to} \quad 12x + 6y = 11 \tag{L}$$
$$\max F(x,y) \quad \text{subject to} \quad 12x + 6y \geq 11 \tag{K}$$

($12x + 6y \geq 11$ is equivalent to $11 - 12x - 6y \leq 0$ if you prefer the inequality in that direction.)

(b) Consider the point $(\hat{x}, \hat{y}) = (\frac{3}{4}, \frac{1}{3})$.
- Verify that $(\hat{x}, \hat{y})$ satisfies the Lagrange conditions associated with problem (L).
- Does $(\hat{x}, \hat{y})$ satisfy the Kuhn–Tucker conditions associated with problem (K)?

In the following, you can use without proof the fact that $(\hat{x}, \hat{y})$ is the *only* point that satisfies the Lagrange conditions associated with problem (L).

Consider now the following optimization problem, with one more constraint than (K):

$$\max F(x,y) \qquad \text{subject to} \qquad 12x + 6y \geq 11 \quad \text{and} \quad x + y \leq 1 \tag{P}$$

(c) Show that if the Kuhn–Tucker conditions associated with (P) hold at an admissible point (x, y) then we cannot have $x + y < 1$.

(d) It is a fact that there is one unique point (x^*, y^*) that satisfies the Kuhn–Tucker conditions, and that both the Lagrange multipliers are > 0.
Show that (x^*, y^*) solves problem (P).

Notes: The solution to follow will direct the inequality constraints to «$\leq$» as the book does. Candidates who choose to do otherwise, would have to get sign conditions appropriate for their problem formulation.
For (d), the committee has to take a stand on whether it warrants any score at all to express familiarity with the extreme value theorem – which does not apply to problem (P).

How to solve: We will soon need the partial derivatives $F'_x(x, y) = 1 - 2x - y$ and $F'_y(x, y) = 1 - x - 2y$.

(a) $\boxed{\text{The answer is negative:}}$ $xF'_x(x, y) + yF'_y(x, y) = x - xy - 2x^2 + y - xy - 2y^2 = x + y - 2xy - 2x^2 - 2y^2$ is not a scaling of $F(x, y)$: it equals $2F(x, y) - x - y$, so for homogeneity, then $(x + y)$ would also have to be a scaling of F. It is not.

(b) We need a Lagrangian $L(x, y) = F(x, y) - \lambda(11 - 12x - 6y)$ (rewriting to get the form with «$\leq$» inequality). Its partial derivatives are $F'_x(x, y) + 12\lambda = 1 - 2x - y + 12\lambda$ and $F'_y(x, y) + 6\lambda = 1 - x - 2y + 6\lambda$. We have $L'_x(\frac{3}{4}, \frac{1}{3}) = 0$ iff $12\lambda = 2 \cdot \frac{3}{4} + \frac{1}{3} - 1 = \frac{9}{6} + \frac{2}{6} - \frac{6}{6} = \frac{5}{6}$ i.e. iff $\lambda = 5/72$. We have $L'_y(\frac{3}{4}, \frac{1}{3}) = 0$ iff $6\lambda = \frac{3}{4} + 2 \cdot \frac{1}{3} - = \frac{9}{12} + \frac{8}{12} \cdot \frac{12}{12} = \frac{5}{12}$ i.e. iff $\lambda = 5/72$. So $\lambda = 5/72$ makes $(\hat{x}, \hat{y})$ a stationary point for L.
- For the Lagrange conditions, the only remaining part is to verify the constraint. $12 \cdot \frac{3}{4} + 6 \cdot 13 = 9 + 2$ equals 11 as it should.
- With the Lagrange conditions satisfied, then the Kuhn–Tucker conditions $\boxed{\text{hold because the multiplier } \lambda = 5/72 \text{ is } \geq 0.}$

Associated to (P) is the new Lagrangian $K(x, y) = L(x, y) - \mu(x + y - 1)$.

(c) Suppose for contradiction that $x + y < 1$. Then $\mu = 0$ and $K = L$. There are two cases to rule out:

- Case $12x + 6y = 11$. This equality together with stationarity of K (with $\mu = 0$), amount to the Lagrange conditions associated to (L), and from the problem text, that implies $(x, y) = (\hat{x}, \hat{y})$. But $\hat{x} + \hat{y} = \frac{3}{4} + \frac{1}{3}$ is not < 1, contrary to assumption (and indeed also violating the constraint).
- Case $12x + 6y > 11$. Then $\lambda = 0$ and the function K equals F. Thus we are looking for stationary points for F, leading to the equation system $2x + y = 1$, $x + 2y = 1$. But when $2x + y = 1$, then (multiply by 6) $12x + 6y = 6$ which is < 11.
  (Note: Of course, you can pick a method to solve that linear equation system. E.g. Cramér: $x = \frac{1}{4-1}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = \frac{1}{3}$, $y = \frac{1}{3}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = \frac{1}{3}$. And $12x + 6y = 6 < 11$ still.)

(d) It suffices to show that K is concave: $K''_{xx} = F''_{xx} + 0 = -2$ which is negative, $K''_{yy} = F''_{yy} + 0 = -2$, $K''_{xy} = F''_{xy} + 0 = -1$, Hessian determinant $= (-2) \cdot (-2) - 1^2 > 0$.