Some key steps in the solution

8.60

a)

Step 1. Find the log likelihood function

Step 2. Using first order condition, get $\widehat{\tau}_{mle} = \overline{X}$

b)

$E(X)$, $E(X^2)$ can be obtained by integration by part.

You can also find $E(X)$, $Var(X)$ of the exponential distribution in the textbook, if you notice it is a special case of Gamma distribution.

$E(\widehat{\tau}_{mle}) = E(\overline{X}) = \frac{1}{n}E(\sum_i X_i) = \frac{1}{n}\sum_i EX_i$

$\overset{\text{identical distribution}}{=} \frac{1}{n}nE(X) = E(X) = \tau.$

So it is unbiased.

$Var(\widehat{\tau}_{mle}) = Var(\overline{X}) = \frac{1}{n^2}Var(\sum_i X_i)$

$\overset{\text{independent}}{=} \frac{1}{n^2}\sum_i Var(X_i) \overset{\text{identical distribution}}{=} \frac{1}{n^2}nVar(X) = \frac{Var(X)}{n}$

$= \frac{1}{n}[E(X^2) - (EX)^2] = \frac{1}{n}[2\tau^2 - \tau^2] = \frac{\tau^2}{n}$

d)

$\log f(x|\tau) = -x/\tau - \log\tau$

$I(\tau) = E[(\frac{\partial \log f(X|\tau)}{\partial\tau})^2] = E\left[(\frac{X}{\tau^2} - \frac{1}{\tau})^2\right] = E\left[\frac{X^2}{\tau^4} + \frac{1}{\tau^2} - \frac{2X}{\tau^3}\right] = \frac{E(X^2)}{\tau^4} + \frac{1}{\tau^2} - \frac{2E(X)}{\tau^3} = \frac{1}{\tau^2}$

Using Cramer-Rao inequality to show $Var(\widehat{\tau}_{mle}) = \frac{\tau^2}{n} \leq Var(T)$, where $T$ is any unbiased estimator of $\tau$.

8.68

a)

$$P(X_1 = x_1, ..X_n = x_n | \sum_i^n X_i = t) = \frac{P(X_1 = x_1, ..X_n = x_n, \sum_i^n X_1 = t)}{P(\sum_i^n X_i = t)}$$

Notice $P(X_1 = x_1, ..X_n = x_n, \sum_i^n X_i = t)$ is the probability of *one* specific outcome where $\sum_i^n x_i = t$

$$P(X_1 = x_1, ..X_n = x_n) = \prod_{i=1}^n \frac{\lambda^{x_i}e^{-\lambda}}{x_i!}$$

Notice $P(\sum_i^n X_i = t)$ is the sum of *all* the probability of the outcomes where $\sum_i^n y_i = t$

$$
\begin{aligned}
P(\sum_i^n X_i &= t) = \sum_{y_1=0}^{t} \sum_{y_2=0}^{t-y_2} ... \sum_{y_n=0}^{t-(y_1+...y_{n-1})} \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y_i!} \\
&= e^{-n\lambda} \lambda^{(y_1+..y_n)} \left\{ \sum_{y_1=0}^{t} \sum_{y_2=0}^{t-y_2} ... \sum_{y_n=0}^{t-(y_1+...y_{n-1})} (\prod_{i=1}^{n} \frac{1}{y_i!}) \right\}
\end{aligned}
$$

To prove it is independent of $\lambda$, just notice $e^{-n\lambda}\lambda^{(y_1+..y_n)}/e^{-n\lambda}\lambda^{(x_1+..x_n)} = 1$

b)

$P(X_1 = x_1, ..X_n = x_n | X_1 = t) = \frac{P(X_1=x_1,..X_n=x_n,X_1=t)}{P(X_1=t)}$

Notice $P(X_1 = x_1, ..X_n = x_n, X_1 = t) = P(X_1 = t, X_2 = x_2...X_n = x_n) = P(X_1 = t)\prod_{i=2}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$

c) and 8. 69, 8. 70

Step 1. Write down the joint probability frequency (density) function

Step 2. Notice it is a product of frequency (density) function $\prod_{i=1}^{n} f(x_i|\theta)$

Step 3. Combining the terms which contain $x_i$

Step 4. Define $T = t(X)$.In the above three cases, $T = \sum_{i=1}^{n} x_i$.

Step 5. Define $g(\theta, T)$ and $h(X)$, and thus use the Factorization Theorem to prove $T$ is sufficient.