Econ4130 10H

Exercises for seminar week 40

Rice chapter 3:

10 + ekstra: Find the regression functions, $E(Y \mid x)$ and $E(X \mid y)$. Are they linear ? Are they homoscedastic or heteroscedastic? [**Hint:** Identify both of the conditional distributions as gamma-distributions.]

18 + extra: Find $E(Y \mid x)$ [**Hint:** For **18b,c,** read **general hints** below.**]**

Rice chapter 4:

81 (see section 2.1.1)

82 (see last paragraph of section 2.1.2)

Problem 2 in postponed exam 2006

[**Note** a printing mistake in question **a**: var(V) should be $\frac{n^2-1}{12}$, not $\frac{(n+1)^2}{12}$ as in the text]

[**Hint** for question **d**: Utilize the mgf of T]

[**Note** also that, because of a bug in Word's printing software, some formulas like, $1+2+\cdots+n$, the $\cdots$ came out like L in the text, and … came out like K, e.g., $X_1,\dots,X_n$, becomes X_1, K, X_n]

General hints: **Integration over non rectangular areas.**

In some of the exercises you will have to calculate double integrals over areas that are not rectangles. Since I didn't have time to talk about that in the lectures, I will give you an example as introduction. There is nothing new involved - only that you need to be a bit careful with the integration limits.

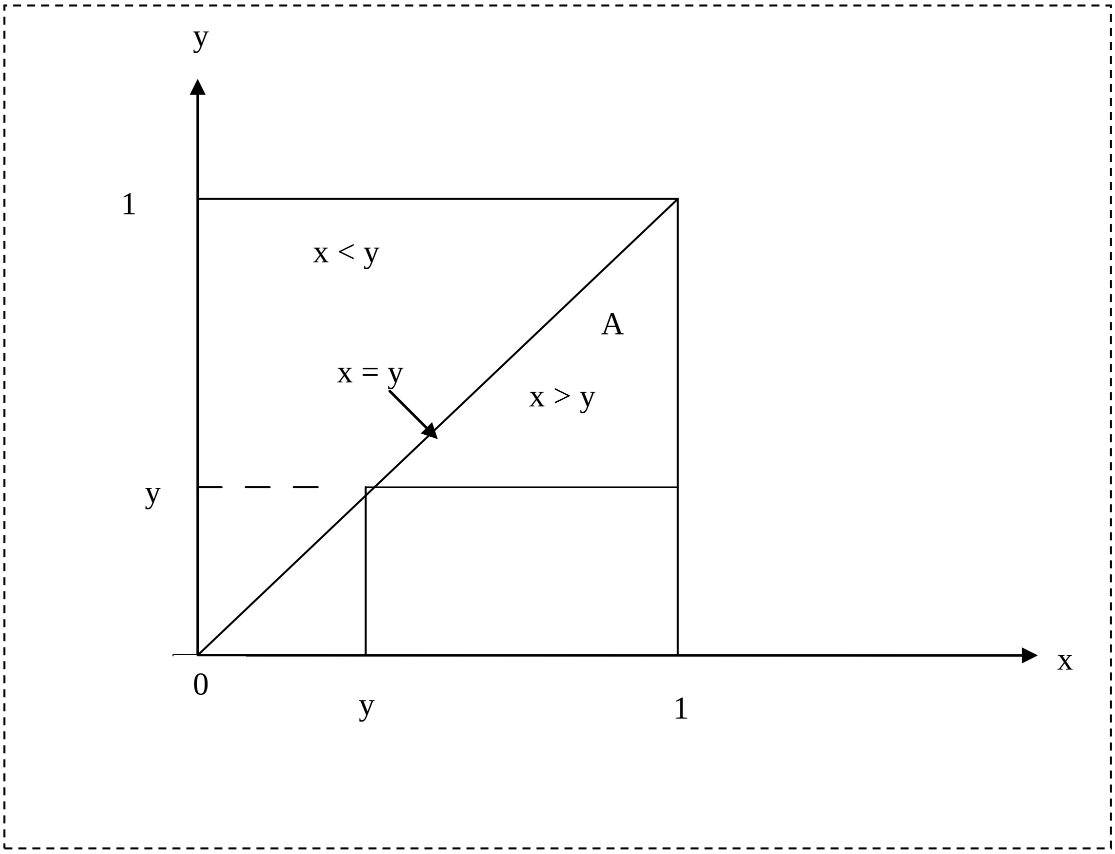
We will look at an example from the lectures. Suppose $(X,Y) \sim f(x,y)$, where the pdf, *f* is

$$f(x,y) = \begin{cases} \frac{12}{7}(x^2+xy) & \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

What is $P(X > Y)$?

Solution: Look at figure 1. What we ask for is the probability that an observation of (X,Y) will be a point falling in the lower triangle called *A* on the figure, consisting of all points (x,y) (within the square of possible observations) where $x > y$.

Figure 1



According to the theory this probability is the volume under the pdf over that area, i.e., the integral over *A* of *f*:

$$P(X > Y) = P((X,Y) \in A) = \iint_A f(x,y)dxdy = \int_0^1 \left[\int_y^1 f(x,y)dx \right] dy$$

Explanation of the inner integral: The inner integral you integrate with respect to *x* while keeping *y* fixed. Now proceed as follows: Fix first a *y* somewhere arbitrary between 0 and 1 on the y-axis (see the figure). Then find out which *x*'s (on the x-axis) are such that (x, y) belongs to *A* for that particular *y*. Looking at the figure we see that all *x* such that

$y \le x \le 1$ satisfy this. Hence the inner integral must be over the interval $[y,1]$, i.e.,
$\int_y^1 f(x,y)dx$, giving

$$\int_y^1 f(x,y)dx = \frac{12}{7}\int_y^1 (x^2 + xy)dx = \frac{12}{7}\Big|_y^1 \left(\frac{1}{3}x^3 + y\frac{1}{2}x^2\right) = \frac{12}{7}\left[\frac{1}{3} + \frac{y}{2} - \frac{y^3}{3} - \frac{y^3}{2}\right] = \frac{12}{7}\left[\frac{1}{3} + \frac{y}{2} - \frac{5}{6}y^3\right] =$$

$$= \frac{2}{7}\left[2 + 3y - 5y^3\right] \qquad \text{for any chosen } y \text{ in the interval } [0,1]$$

Having found the inner integral as a function of *y*, we can now integrate that function over all values of *y* where the pdf, *f*, can be > 0, i.e., over the interval $[0,1]$.
Hence

$$P(X > Y) = \iint_A f(x,y)dxdy = \int_0^1 \left[\int_y^1 f(x,y)dx\right]dy = \frac{2}{7}\int_0^1 \left(2 + 3y - 5y^3\right)dy =$$

$$= \frac{2}{7}\Big|_0^1 \left[2y + \frac{3}{2}y^2 - \frac{5}{4}y^4\right] = \frac{2}{7}\left[2 + \frac{3}{2} - \frac{5}{4}\right] = \frac{2}{7}\cdot\frac{8+6-5}{4} = \frac{9}{14}$$

You could also, of course, have integrated the other way, with respect to *y* first (fixing *x* on the x-axis), and then with respect to *x*. Do that yourself for practice, and check that you get the same answer. You may also practice on Rice exercise 3:8a. for which I give the answers (and hope that I calculated correctly (!)):
$P(X > Y) = 1/2, \quad P(X + Y \le 1) = 3/14, \quad P(X \le \tfrac{1}{2}) = 2/7$.