Econ 4130 Exam 2009 H

ANSWERS

Settet er inndelt i 9 deloppgaver, a,b,c,… som alle anbefales å telle likt for å gjøre det litt lettere å stå. Svar er gitt i << ….. >>.

Problem 1

a. Suppose that the continuous random variable (rv) Y has the following cumulative distribution function (cdf)

$$F(y) = \begin{cases} 1 - e^{-\beta y^2} & \text{for y} > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\beta > 0$ is a parameter.

Show that

(i) the probability density function (pdf) is given by

$$f(y) = \begin{cases} 2\beta y e^{-\beta y^2} & \text{for y} > 0 \\ 0 & \text{otherwise} \end{cases}$$

(ii) the median, *m*, of *Y* is given by, $m = \sqrt{\dfrac{\ln(2)}{\beta}}$, and

(iii) the median, m_1, of $1/Y$ is given by $m_1 = \sqrt{\dfrac{\beta}{\ln(2)}}$.

<< **Answer:** **(i)** For $y > 0$, $f(y) = F'(y) = -e^{-\beta y^2}(-2\beta y) = 2\beta y e^{-\beta y^2}$.

(**ii**) $F(m)=1/2$ gives $e^{-\beta m^2}=1/2$, or $m^2=\dfrac{\ln(2)}{\beta}$. Since $m>0$, we get the answer.
(**iii**) $1/2=P(1/Y\le m_1)=P(1/m_1\le Y)$ implies that $1/m_1=m$ (since $P(Y=m)=0$), or $m_1=1/m$. >>

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b. (**i**) Show that $Z=\beta Y^2$ is exponentially distributed with parameter 1 (in short $Z\sim\exp(1)$), and (**ii**) calculate $P\left(Y>1/\sqrt{\beta}\right)$.

<< **Answer:** (**i**) $P(Z\le z)=P(\beta Y^2\le z)=P(Y^2\le z/\beta)$. Since *Y* must be > 0, we get
$P(Z\le z)=P\left(Y\le\sqrt{z/\beta}\right)=F\left(\sqrt{z/\beta}\right)=1-e^{-z}$.
(**ii**) $P\left(Y>1/\sqrt{\beta}\right)=1-F\left(1/\sqrt{\beta}\right)=e^{-1}=0.368$. >>

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c. Show that (**i**) $E(Y)=\dfrac{1}{\sqrt{\beta}}\cdot\dfrac{\sqrt{\pi}}{2}$, and (**ii**) $E\left(\dfrac{1}{Y}\right)=E(Y^{-1})=\sqrt{\beta}\cdot\sqrt{\pi}$.

[**Hint:** Utilize the relationship between *Y* and *Z*, i.e., $Y=\dfrac{1}{\sqrt{\beta}}Z^{\frac{1}{2}}$, and the well known moment formula for gamma distributed variables:
If $U\sim\Gamma(\alpha,\lambda)$-distributed, then $E\left(U^r\right)=\dfrac{\Gamma(\alpha+r)}{\lambda^r\Gamma(\alpha)}$, which is valid for any real $r>-\alpha$. Notice in particular that the formula holds for negative *r*'s, as long as $r>-\alpha$ is fulfilled.]

<< **Answer:** Since $Z\sim\exp(1)=\Gamma(1,1)$, we have $E(Z^r)=\Gamma(1+r)$ for $r>-1$. Now
$Z=\beta Y^2\Rightarrow Y=\dfrac{1}{\sqrt{\beta}}Z^{\frac{1}{2}}$ and $Y^{-1}=\sqrt{\beta}Z^{-\frac{1}{2}}$. Hence

$$E(Y) = \frac{1}{\sqrt{\beta}} E\left(Z^{\frac{1}{2}}\right) = \frac{1}{\sqrt{\beta}} \Gamma\left(\frac{3}{2}\right) = \frac{1}{\sqrt{\beta}} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\beta}} \cdot \frac{\sqrt{\pi}}{2}$$

$$E(Y^{-1}) = \sqrt{\beta} \cdot E\left(Z^{-\frac{1}{2}}\right) = \sqrt{\beta} \cdot \Gamma\left(\frac{1}{2}\right) = \sqrt{\beta} \cdot \sqrt{\pi}$$ >>

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d. **Introduction.** The lifetime of a certain unit used in electronic equipments is defined as the time it takes until breakdown under continuous use. As part of the quality control the producer of the unit yearly draws a sample of n = 10 units from the production and puts them to constant use until breakdown. The purpose, among other things, is to check if the average lifetime of the units in the production is up to the prescribed standard of minimum 3.5 months.

Table 1 shows the lifetimes from one such test. Earlier experience shows that the distribution F in section **a.** fits well for lifetimes of this type. Let Y_i denote the lifetime i in the sample, $i = 1, 2, \ldots, n$, where $n = 10$ in the sample. As our model we therefore assume: $Y_1, Y_2, \ldots, Y_n$ are *iid*[1] with Y_i distributed as F in section **a**, and where the true value of the parameter β is unknown for the present sample.

Table 1 **Observed lifetimes (in months)**

i	1	2	3	4	5	6	7	8	9	10
Y_i	5.2	3.1	9.9	3.0	7.4	4.7	4.5	3.8	4.2	1.0

$$\sum_i Y_i = 46.8 \qquad \sum_i Y_i^2 = 273.84$$

Question.

(i) Show that the maximum likelihood estimator (mle) of β is

$$\hat{\beta} = \frac{n}{\sum_{i=1}^{n} Y_i^2}$$

(ii) Find the moment method estimator (mme) for β [**Hint**: Use section **c(i)**]

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[1] *iid* means independent and identically distributed.

(iii) The parameter β is only of secondary interest in this situation. The parameter of prime interest is the mean lifetime of units in the production, $\mu = E(Y_i)$. What are the mle and the mme for μ? Calculate the estimates from the data.

<<**Answer.** (**i**) The log-likelihood is $\ell(\beta) = n\ln(2) + n\ln(\beta) + \sum_i y_i - \beta\sum_i y_i^2$, with derivative $\ell'(\beta) = \frac{n}{\beta} - \sum_i y_i^2 = 0$ for $\beta = \frac{n}{\sum_i y_i^2}$. Since $\ell''(\beta) = -\frac{n}{\beta^2} < 0$, the mle becomes as given in the question.

(**ii**). Section **c(ii)** gives for the first moment, $\mu_1 = E(Y_i) = \frac{1}{\sqrt{\beta}} \cdot \frac{\sqrt{\pi}}{2} \Leftrightarrow \beta = \frac{\pi}{4\mu_1^2}$,

and the mme, $\tilde{\beta} = \frac{\pi}{4\bar{Y}^2}$

(**iii**) The invariance property for mle's gives the mle, $\hat{\mu} = \frac{1}{\sqrt{\hat{\beta}}} \cdot \frac{\sqrt{\pi}}{2}$ with estimate $\hat{\mu}_{obs} = 4.64$. ($\hat{\beta}_{obs} = 0.03652$). The mme is $\tilde{\mu} = \tilde{\mu}_1 = \bar{Y}$ with estimate $\tilde{\mu}_{obs} = 4.68$ ($\tilde{\beta}_{obs} = 0.035859$).
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e. (**i**) Develop the formula for an approximate 95% confidence interval (CI) for μ based on the asymptotic properties of $\hat{\beta}$.

[**Hint:** Show first that the Fisher information for one observation is $I(\beta) = \frac{1}{\beta^2}$. Then develop the formula for an approximate 95% CI for β. Finally transform this interval to the corresponding CI for μ, using the relationship between μ and β.]

(**ii**) Calculate (using the data) the CI for μ.

<<**Answer.** **(i)** We need the Fisher information for one observation,

$I(\beta) = -E\left(\frac{\partial^2}{\partial\beta^2}\ln(f(Y \mid \beta)\right)$. Write short $f = f(Y \mid \beta)$. We get

$$\ln(f) = \ln(2) + \ln(\beta) + \ln(Y) - \beta Y^2 \;\Rightarrow\; \frac{\partial}{\partial\beta}\ln(f) = \frac{1}{\beta} - Y^2 \;\Rightarrow\; \frac{\partial^2}{\partial\beta^2}\ln(f) = -\frac{1}{\beta^2}$$

Hence $I(\beta) = -E\left(-\frac{1}{\beta^2}\right) = \frac{1}{\beta^2}$. The mle-theory then gives

$$\sqrt{nI(\beta)}(\hat{\beta} - \beta) = \sqrt{n}\,\frac{\hat{\beta} - \beta}{\beta} \xrightarrow[n\to\infty]{D} N(0,1)\,.$$

Using Slutsky's theorem and the fact that the mle $\hat{\beta}$ is consistent, we also get $\sqrt{n}\,\frac{\hat{\beta} - \beta}{\hat{\beta}} \xrightarrow[n\to\infty]{D} N(0,1)$, from which we obtain

$$P(L \le \beta \le U) = P\left(\hat{\beta} - 1.96\frac{\hat{\beta}}{\sqrt{n}} \le \beta \le \hat{\beta} + 1.96\frac{\hat{\beta}}{\sqrt{n}}\right) \approx 0.95\,,$$

here *L* and *U* denote the lower and upper limits of the CI. Since $\mu = \frac{1}{\sqrt{\beta}}\cdot\frac{\sqrt{\pi}}{2}$ is a decreasing function of β, we get the approximate 95% CI for μ as

$$\left(\frac{1}{\sqrt{U}}\cdot\frac{\sqrt{\pi}}{2},\; \frac{1}{\sqrt{L}}\cdot\frac{\sqrt{\pi}}{2}\right), \quad \text{since}$$

$$P\left(\frac{1}{\sqrt{U}}\cdot\frac{\sqrt{\pi}}{2} \le \mu \le \frac{1}{\sqrt{L}}\cdot\frac{\sqrt{\pi}}{2}\right) = P(L \le \beta \le U) \approx 0.95\,.$$

(ii) $\hat{\beta}_{obs} = 0.03652$ gives $(0.01388,\ 0.05916)$ for β, which leads to $(3.64, 7.52)$ for μ.
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f. In this section we shall develop an exact 95% CI for μ:

(i) Let $Z_i = \beta Y_i^2$ for $i = 1, 2, \ldots, n$. Then we have that $Z_1, Z_2, \ldots, Z_n$ are *iid* with $Z_i \sim \exp(1)$ (according to section **b)**. Utilizing the moment generating function (mgf) for the exp(1)-distribution, show that $V = \beta \sum_{i=1}^{n} Y_i^2 = \sum_{i=1}^{n} Z_i$ is $\Gamma(n,1)$-distributed.

(ii) Show that V can be written $V = n\frac{\beta}{\hat{\beta}} = n\frac{\hat{\mu}^2}{\mu^2}$, where $\hat{\beta}, \hat{\mu}$ are the mle's for β, μ respectively, and $n = 10$. Use this and the result in (**i**) to derive an exact 95% CI for μ. [**Hint.** Start with a statement like $P(c_1 \le V \le c_2) = 0.95$ where c_1, c_2 are suitable quantiles taken from table 2 below.]

(iii) Calculate the exact CI for μ from the data and comment on the difference (if any) from the approximate CI for μ calculated in section **e**.

Table 2 **Quantiles of the $\Gamma(10,1)$- distribution.**
(The p-quantile, v_p, is defined by $P(V \le v_p) = p$)

p	0.01	0.025	0.05	0.95	0.975	0.99
v_p	4.13	4.80	5.43	15.71	17.08	18.78

<< **Answer.** (**i**) The mgf for $\exp(1) = \Gamma(1,1)$ is $M(t) = \frac{1}{1-t}$. Hence the mgf for V is

$$M(t)^n = \left(\frac{1}{1-t}\right)^n$$ which is the mgf of $\Gamma(n,1)$.

(ii) Since $\sum_i Y_i^2 = \frac{n}{\hat{\beta}}$, we have $V = n\frac{\beta}{\hat{\beta}}$. Now, $\mu = \frac{\sqrt{\pi}}{2\sqrt{\beta}}$ implies, $\beta = \frac{\pi}{4\mu^2}$ and $\hat{\beta} = \frac{\pi}{4\hat{\mu}^2}$. Hence $V = n\frac{\hat{\mu}^2}{\mu^2}$.

$$0.95 = P\left(c_1 \le n\frac{\hat{\mu}^2}{\mu^2} \le c_2 \right) = P\left(\frac{n}{c_2}\hat{\mu}^2 \le \mu^2 \le \frac{n}{c_1}\hat{\mu}^2 \right) = P\left(\sqrt{\frac{n}{c_2}}\hat{\mu} \le \mu \le \sqrt{\frac{n}{c_1}}\hat{\mu} \right)$$

$$= P\left(\sqrt{\frac{10}{17.08}}\hat{\mu} \le \mu \le \sqrt{\frac{10}{4.8}}\hat{\mu} \right) = P\left(0.765 \cdot \hat{\mu} \le \mu \le 1.443 \cdot \hat{\mu}\right)$$

(iii) $\hat{\mu} = 4.64$ gives the exact 95% CI for μ: $(0.765 \cdot \hat{\mu},\ 1.443 \cdot \hat{\mu}) = (3.55,\ 6.70)$. The approximate CI was $(3.64, 7.52)$, which indicates that $n = 10$ is somewhat small for the asymptotic mle-theory to work properly in this situation.

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Problem 2

a. Let X be a continuous random variable with pdf

$$f(x) = \begin{cases} \frac{1}{x} & \text{for } 1 \le x \le e \quad (= 2.71828...) \\ 0 & \text{otherwise} \end{cases}$$

(i) Find the cdf, $F(x) = P(X \le x)$ and make a sketch of its graph.

(ii) Show that for any real $r \ne 0$

$$E(X^r) = \frac{e^r - 1}{r}$$

<< **Answer.** **(i)** For $1 \le x \le e$, $F(x) = \int_1^x \frac{1}{u} du = \Big|_1^x \ln(u) = \ln(x)$. Thus

$$F(x) = \begin{cases} 0 & \text{for } x < 1 \\ \ln(x) & \text{for } 1 \le x \le e \\ 1 & \text{for } x > e \end{cases}$$

Graph = ….

(ii) $$E(X^r) = \int_1^e x^r \frac{1}{x} dx = \int_1^e x^{r-1} dx = \Big|_1^e \frac{1}{r} x^r = \frac{e^r - 1}{r}$$

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b. Let Y be another continuous random variable related to X in such a way that the conditional distribution of Y given $X = x$, is exponential with parameter x, i.e., such that the conditional pdf is

$$h(y \mid x) = \begin{cases} xe^{-xy} & \text{for } 0 < y < \infty, \quad 1 \le x \le e \\ 0 & \text{otherwise} \end{cases}$$

(i) Using formulas for expectation and variance in the exponential distribution, write up expressions for $E(Y \mid x)$ and $\text{var}(Y \mid x)$.

(ii) Find $E(Y)$ and $\text{var}(Y)$.

<< **Answer.** **(i)** $E(Y \mid x) = \dfrac{1}{x}$, $\quad \text{var}(Y \mid x) = \dfrac{1}{x^2}$

(ii)

$$E(Y) = E\big(E(Y \mid X)\big) = E(X^{-1}) = \frac{e^{-1} - 1}{-1} = 1 - e^{-1} \quad (= 0.632...)$$

$$\text{var}(Y) = E\big(\text{var}(Y \mid X)\big) + \text{var}\big(E(Y \mid X)\big) = E\big(X^{-2}\big) + \text{var}(X^{-1}) =$$

$$= 2E(X^{-2}) - \big(E(X^{-1})\big)^2 = 2\frac{e^{-2} - 1}{-2} - (1 - e^{-1})^2 =$$

$$= 1 - e^{-2} - 1 + 2e^{-1} - e^{-2} = 2(e^{-1} - e^{-2}) \quad (= 0.465...)$$

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c. Let 0.23, 0.70, 0.49 be three independent observations drawn from the uniform distribution over the interval (0, 1).
(i) Transform these observations so that the transformed values can be considered as three independent observations of X.

(**ii**) Can you think of a way to simulate joint observations of (X, Y) with joint pdf, $g(x, y) = h(y \mid x) f(x)$ based on observations drawn from the (0, 1)-uniform distribution?

<< **Answer.** (**i**) If $U \sim$ uniform$(0,1)$, $X = F^{-1}(U) \sim F$. Now

$x = F^{-1}(u) \Leftrightarrow u = F(x) = \ln(x) \Leftrightarrow x = e^u$. Hence $X = e^U \sim F$ and $\left(e^{0.23}, e^{0.7}, e^{0.49}\right) = (1.26, 2.01, 1.63)$ can be considered as drawn from *F*.

(**ii**) Generate first an observation of *X*, $x_{obs} = e^{u_{obs}}$, where u_{obs} is an observation of $U \sim$ uniform$(0,1)$. Then the corresponding *Y* is distributed as exp(x_{obs}) with (conditional) cdf, $H(y) = 1 - e^{x_{obs} y}$, and inverse, $H^{-1}(v) = -\frac{1}{x_{obs}} \ln(1 - v)$ for $0 < v < 1$.

If $V \sim$ uniform$(0,1)$, then $Y = H^{-1}(V) \sim H$, so we get an observation of $(Y \mid x_{obs})$ as $y_{obs} = -\frac{1}{x_{obs}} \ln(1 - v_{obs})$ where v_{obs} is an observation of *V*. Hence (x_{obs}, y_{obs}) is an observation of (X, Y).

In other words: If U, V are independent and both $\sim$ uniform$(0,1)$, then $(X, Y) = \left(e^U, \; -e^{-U} \ln(1 - V)\right)$ has the right joint distribution.

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