Econ 4130 Regular Exam 2021H

ANSWERS

Settet er inndelt i 9 deloppgaver, A,B,C,… som alle anbefales å telle likt for å gjøre det litt lettere å stå. Svar er gitt i <<< ….. >>>.

Problem 1

A. Let X be exponentially distributed with parameter, $\lambda = 1$ and with probability density function (pdf)

$$f_X(x) = \begin{cases} e^{-x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases} \qquad \text{(Written in short: } X \sim \text{Exp(1)}\text{),}$$

i. Derive the cumulative distribution function (cdf), $F_X(x) = P(X \le x)$, and calculate $P(1 \le X \le 2)$.

ii. Show that $E(X^k) = k!$ for $k = 1, 2, 3, \ldots$ by utilizing the relationship to the gamma distribution or to the gamma function.

[Hint. You may, for example, use (here or later in the exam set without proving) that, if V is gamma distributed with parameters (α, λ), then

$$E(V^r) = \frac{\Gamma(\alpha + r)}{\lambda^r \, \Gamma(\alpha)}, \quad \text{for any real } r > -\alpha \text{.} \quad \textbf{]}$$

<<< **Answer: i):** If $x \le 0$, then $f_X(x) = 0 \Rightarrow F_X(x) = \int_{-\infty}^{x} 0 \, du = 0$.

If $x > 0$, then $F_X(x) = \int_{-\infty}^{x} f_X(u)du = \int_{0}^{x} e^{-u} du = \Big|_0^x (-e^{-u}) = 1 - e^{-x}$. So,

$$F_X(x) = \begin{cases} 1 - e^{-x} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

ii): Directly from the gamma function: $E(X^k) = \int_0^\infty x^k e^{-x} dx = \int_0^\infty x^{k+1-1} e^{-x} dx = \Gamma(k+1) = k!$

Or, via the gamma distribution: $X \sim \exp(1) = \Gamma(1,1) \Rightarrow E(X^k) = \frac{\Gamma(1+k)}{1^k \Gamma(1)} = \Gamma(1+k) = k!$

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B. Let X be as in **A** and Y another continuous random variable (rv) such that the conditional distribution of Y, given that $X = x$ is fixed, is a gamma distribution with shape $\alpha = x^2$ and scale $\lambda = 1$ (in short: $(Y \mid x) \sim \Gamma(x^2, 1)$).

i. Explain why the regression function of Y with respect to X is given by $E(Y \mid x) = x^2$. Is the regression homoscedastic or heteroscedastic? Give a reason for your answer.

ii. Show that $E(Y) = 2$ and $\text{var}(Y) = 22$.

[Hint. Use, for example, the iterated (Rice: "total") expectation theorem. **]**

<<< Answer:

i): If $V \sim \Gamma(\alpha, \lambda)$, we know from the curriculum that $E(V) = \frac{\alpha}{\lambda}$ and $\text{var}(V) = \frac{\alpha}{\lambda^2}$.

Hence, $(Y \mid x) \sim \Gamma(x^2, 1) \Rightarrow E(Y \mid x) = x^2$ and $\text{var}(Y \mid x) = x^2$.

Since $\text{var}(Y \mid x) = x^2$ is not a constant function of x, the regression is heteroscedastic.

ii): $E(Y) = E\left[E(Y \mid X)\right] = E\left(X^2\right) \overset{A}{=} 2! = 2$

$$\text{var}(Y) = E\left[\text{var}(Y \mid X)\right] + \text{var}\left[E(Y \mid X)\right] = E\left(X^2\right) + \text{var}\left(X^2\right) =$$

$$= E\left(X^2\right) + E\left(X^4\right) - \left(E(X^2)\right)^2 = 2! + 4! - (2!)^2 = 2 + 24 - 4 = 22$$

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C. **i.** Show that $\text{cov}(X, Y) = 4$, and calculate the correlation coefficient, $\rho = \text{corr}(X, Y)$.

[Hint. Remember the formula, $E\left(XY\right) = E\left[X \cdot E(Y \mid X)\right]$, that you may use without proof. **]**

ii. Considering $(X, Y)'$ a random vector, write up the covariance matrix of $(X, Y)'$, i.e., find $\text{cov}\begin{pmatrix} X \\ Y \end{pmatrix}$.

iii. Let W be a random vector of any dimension, and c a constant vector of same dimension as W.

Explain why $\text{cov}(W + c) = \text{cov}(W)$.

<<< Answer:

i): $\text{cov}(X, Y) = E\left(XY\right) - E\left(X\right) \cdot E\left(Y\right)$, where

$E(XY) = E\left[X \cdot E(Y \mid X)\right] = E\left[X \cdot X^2\right] = E\left[X^3\right] = 3! = 6$

Hence, $\mathrm{cov}(X,Y) = 6 - 1 \cdot 2 = 4$, and the correlation,

$$\rho = \frac{\mathrm{cov}(X,Y)}{\sqrt{\mathrm{var}(X)} \cdot \sqrt{\mathrm{var}(Y)}} = \frac{4}{\sqrt{22}} = 0.853...$$

ii): The covariance matrix is $\mathrm{cov}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \mathrm{var}(X) & \mathrm{cov}(X,Y) \\ \mathrm{cov}(X,Y) & \mathrm{var}(Y) \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 4 & 22 \end{pmatrix}$

iii): The definition of the covariance matrix is, $\mathrm{cov}(W) = E\left[\left(W - E(W)\right) \cdot \left(W - E(W)\right)'\right]$.

Since $E\left(W + c\right) = E\left(W\right) + c$, we get

$$\mathrm{cov}(W + c) = E\left[\left(W + c - E(W) - c\right) \cdot \left(W + c - E(W) - c\right)'\right] = E\left[\left(W - E(W)\right) \cdot \left(W - E(W)\right)'\right] = \mathrm{cov}(W)$$

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D. In this section we will prove that, in the present model, there exist constants, *a* and *b*, and a random error term, *u*, such that $Y = a + bX + u$, where *u* and *X* are uncorrelated and $E(u) = 0$.

Put $u = Y - a - bX$ and show that the two equations, $E(u) = 0$ and $E(u \cdot X) = 0$, lead to setting $a = -2$ and $b = 4$, and that these two values for *a* and *b* imply that *u* and *X* are uncorrelated and $E(u) = 0$.

<<< **Answer:**

Requiring that $E(u) = 0$, makes $\mathrm{cov}(u, X) = 0$ equivalent with $E(uX) = 0$, since $\mathrm{cov}(u, X) = E(uX) - E(u) \cdot E(X)$.

Since $u = Y - a - bX$, we have $E(u) = 0 \Leftrightarrow E(Y) - a - bE(X) = 0 \Leftrightarrow 2 = a + b \cdot 1 = a + b$, and

$$E(uX) = 0 \Leftrightarrow E\left((Y - a - bX)X\right) = 0 \Leftrightarrow E\left(XY\right) - aE(X) - bE(X^2) = 0$$
$$\Leftrightarrow 6 - a - 2b = 0$$

Hence, we have the two equations

$$\begin{matrix} a + 2b = 6 \\ a + b = 2 \end{matrix} \Leftrightarrow b = 4 \text{ and } a = -2$$

On the other hand, putting $u = Y + 2 - 4X$ implies clearly $E(u) = 0$ and $E(uX) = 0$, and therefore also $\mathrm{cov}(u, X) = 0$.

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E. We want to predict *Y* based on an observation of *X*. Two predictors have been suggested: $\hat{Y}_1 = h_1(X) = X^2$ and $\hat{Y}_2 = h_2(X) = a + bX$, where *a*, *b* are given in **D** where $u = Y - a - bX$, $E(u) = 0$, and $\text{cov}(u, X) = 0$.

For any predictor, $\hat{Y} = h(X)$, the mean squared error (*MSE*) is defined as $MSE(\hat{Y}) = E\left[(Y - \hat{Y})^2\right]$.

i. Explain why the *MSE* of $\hat{Y}_2$ must be the same as $\text{var}(u)$.

ii. Show that the *MSE* of both the suggested predictors are, $MSE(\hat{Y}_1) = 2$ and $MSE(\hat{Y}_2) = 6$. Comment briefly on the result.

> **[Hint:** There are several approaches that works in **ii**. In some approaches you may need the following result that you are allowed to use here without formal proof:
>
> If $g(x, y)$ is a function of two variables and $g(X, Y)$ the resulting rv, then fixing $X = x$ implies that $E\left[g(X,Y) \mid X = x\right] = E\left[g(x,Y) \mid X = x\right]$.
>
> In addition, especially for $\hat{Y}_1$, it may be a good idea to exploit the theorem of total (iterated) expectation. **]**

<<< Answer:

i): Since $u = Y - a - bX$, we get

$$MSE(\hat{Y}_2) = E\left[(Y - \hat{Y}_2)^2\right] = E\left[(Y - a - bX)^2\right] = E(u^2) = \text{var}(u) \quad \text{since} \quad E(u) = 0.$$

ii): We find, using $a = -2$ and $b = 4$ from **D**,

$$MSE(\hat{Y}_2) = \text{var}(u) = \text{var}(Y + 2 - 4X) = \text{var}(Y - 4X) = \text{var}\left[(-4, 1)\begin{pmatrix} X \\ Y \end{pmatrix}\right] =$$

$$= (-4, 1)\left[\text{cov}\begin{pmatrix} X \\ Y \end{pmatrix}\right]\begin{pmatrix} -4 \\ 1 \end{pmatrix} = (-4, 1)\begin{pmatrix} 1 & 4 \\ 4 & 22 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \end{pmatrix} = (-4, 1)\begin{pmatrix} 0 \\ 6 \end{pmatrix} = 6$$

(Some candidates would use the usual scalar calculation of $\text{var}(Y + 2 - 4X)$ from the basic course, which should be ok.)

For $\hat{Y}_1$ we have

$$MSE(\hat{Y}_1) = E\left[(Y - \hat{Y}_1)^2\right] = E\left[(Y - X^2)^2\right] = E\left[E\left((Y - X^2)^2 \mid X\right)\right].$$

The last expression has the form $Eg(X)$ where the underlying function of *x* is

$$g(x) = E\left[\left(\left(Y - X^2\right)^2 \mid X = x\right)\right] = E\left[\left(\left(Y - x^2\right)^2 \mid X = x\right)\right] = E\left[\left(\left(Y - E(Y \mid x)\right)^2 \mid X = x\right)\right] = \text{var}(Y \mid x) = x^2$$

Hence,

$$MSE(\hat{Y}_1) = Eg(X) = E(X^2) = 2$$

[An alternative and acceptable approach, that some candidates may do, is straightforward calculation:

$$MSE(\hat{Y}_1) = E\left[(Y - X^2)^2\right] = E\left[Y^2 - 2YX^2 + X^4\right] = \text{var}(Y) + \left(E(Y)\right)^2 - 2E\left(YX^2\right) + E\left(X^4\right)$$

Now, $E\left[YX^2\right] = E\left[E\left(YX^2 \mid X\right)\right] = E\left[X^2 E(Y \mid X)\right] = E\left[X^2 X^2\right] = E\left[X^4\right] = 4!$ and, therefore,

$$MSE(\hat{Y}_1) = \text{var}(Y) + \left(E(Y)\right)^2 - 2E\left(YX^2\right) + E\left(X^4\right) = 22 + 4 - 2 \cdot 4! + 4! = 26 - 24 = 2 \quad]$$

The result, $MSE(\hat{Y}_1) = 2$ and $MSE(\hat{Y}_2) = 6$, indicates that $\hat{Y}_1$ is substantially better (in the *MSE* sense) than the linear predictor $\hat{Y}_2$. This should not be a surprise since the best predictor of *Y* in the *MSE* sense and when the distribution of *Y* is known (being the conditional $f(y \mid x)$ in this case), is the expected value $E(Y \mid x) = x^2$.

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Problem 2

The data[1] in **table 1** are $n = 31$ strength measurements of a type of glass used in airplane windows. We will not go into details about how this is measured, but assume (initially) that the data are observations of *n* independent and identically distributed (iid) random variables (rv), $Y_1, Y_2, \ldots, Y_n$.

Note that the observations in table 1 are sorted in ascending order and do not show the original random order of the observations. So, for example, the first observation, y_1, is not necessarily 18.8, as in the table, but may be any of the numbers.

Table 1 Window strength measurements (sorted in ascending order)

18.8	20.8	21.7	23.0	23.2	24.1	24.3	25.5	25.5	25.8	26.7
26.8	26.8	27.1	27.7	29.9	31.1	33.2	33.7	33.8	33.9	34.8
35.8	35.9	37.0	37.1	37.1	39.6	44.0	45.3	45.4		

Number of observations: $n = 31$. Sample mean, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i = 30.82...$

A. **i.** Let the random variable (rv) *X* be the number of observations less or equal to 30 in a random sample of 31 observations of window strengths, observed under the same condition as in the introduction. Find the observed value of *X* (X_{obs}) based on the

[1] The data are taken from: Fuller, E.J., Frieman, S., Quinn, J., Quinn, G. and Carter, W. (1994) Fracture Mechanics Approach to the Design of Glass Aircraft Windows: A Case Study. Proceeding SPIE, 2286, 419-430.

observations in **table 1**. Explain why *X* is a binomially distributed rv under the conditions described in the introduction.

ii. Develop an approximate 95% confidence interval (CI) for *p* based on *X* (assuming $n = 31$ sufficiently large). If you need Slutsky's lemma in your argument, explain how it is used. Calculate the observed CI.

[Hint. You may take as facts (without proof) that
$\frac{X - np}{\sqrt{np(1-p)}} \overset{D}{\underset{n\to\infty}{\to}} Z \sim N(0,1)$,
and that $\hat{p} = \frac{X}{n}$ is a consistent estimator of *p*. **]**

<<< Answer:

i. Since the observations in table 1 are ordered, we count $X_{obs} = 16$. Each measurement represents a trial. In each trial we observe if the observation is ≤ 30 ("success") or not. The iid assumption for $Y_1, Y_2, \ldots, Y_n$ implies that the probability of "success" $P(Y_i \leq 30)$ is the same for all measurements and that the outcomes of the *n* trials are independent. Hence, the number of "successes", *X*, must be binomially $(31, p)$ distributed.

ii. Dividing by *n* in the numerator and the denominator gives

$$U_n = \frac{X - np}{\sqrt{np(1-p)}} = \frac{\hat{p} - p}{\sqrt{p(1-p)/n}} = \frac{\hat{p} - p}{\sqrt{p(1-p)}}\sqrt{n} \overset{D}{\underset{n\to\infty}{\to}} Z \sim N(0,1)$$

Slutsky's lemma allows us to replace *p* by $\hat{p}$ in the denominator without changing the limit distribution since, by Slutsky's lemma,

$$\frac{\hat{p} - p}{\sqrt{\hat{p}(1-\hat{p})}}\sqrt{n} = \frac{\sqrt{p(1-p)}}{\sqrt{\hat{p}(1-\hat{p})}} \cdot \frac{\hat{p} - p}{\sqrt{p(1-p)}}\sqrt{n} = A_n U_n \overset{D}{\underset{n\to\infty}{\to}} Z \sim N(0,1)$$

where $A_n = \frac{\sqrt{p(1-p)}}{\sqrt{\hat{p}(1-\hat{p})}} \overset{P}{\underset{n\to\infty}{\to}} 1$ because of the continuity of the function

$$h(x) = \frac{\sqrt{p(1-p)}}{\sqrt{x(1-x)}}.$$

Hence, for large *n* (assuming $n = 31$ sufficiently large)

$$P\left(-1.96 \leq \frac{\hat{p} - p}{\sqrt{\hat{p}(1-\hat{p})}}\sqrt{n} \leq 1.96\right) \approx P\left(-1.96 \leq Z \leq 1.96\right) = 0.95$$

The first probability can be written

$$P\left(p \in \left\{ \hat{p} \pm (1.96)\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right\} \right) \approx 0.95 \text{ showing the CI.}$$

Observed CI:

$$\left\{ \hat{p} \pm (1.96)\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \right\}_{obs} = \left\{ 0.516 \pm (1.96)\sqrt{\frac{0.516 \cdot (1-0.516)}{31}} \right\} = \{0.516 \pm (1.96)(0.090)\} =$$
$$= (0.340,\ 0.692)$$

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B. Now, let *Y* be Weibull distributed with two parameters $b > 0$ (scale) and $d > 0$ (shape), (in short: $Y \sim \text{Weibull}(b, d)$). The cdf of *Y* is then

$$F_Y(y) = P(Y \le y) = \begin{cases} 1 - e^{-\left(\frac{y}{b}\right)^d} & \text{for } y > 0 \\ 0 & \text{for } y \le 0 \end{cases}$$

i. If $c > 0$ is a constant, show that $Y \sim \text{Weibull}(b, d) \Rightarrow cY \sim \text{Weibull}(c \cdot b,\ d)$.

ii. Another interpretation of the parameter *b* is that *b* represents a certain quantile (Rice: "percentile") in the distribution. Show that $P(Y \le b)$ has the same value in all Weibull distributions. Which quantile does *b* represent?

iii. Show that *Y* can be written $Y = bV^{\frac{1}{d}}$, where $V \sim \text{Exp}(1)$ (see Problem 1), and show that this implies that

$$E(Y) = b \cdot \Gamma\left(1 + \frac{1}{d}\right)$$

[Hint: Show first that $V = \left(\frac{Y}{b}\right)^d \sim \text{Exp}(1)$ **]**

<<< Answer:

i. Let $V = cY$. Then *V* varies on the positive axis, and the cdf becomes,
$F_V(v) = P(Y \le v/c) = 0$ for $v \le 0$.

For $v > 0$, we get $F_V(v) = P\left(Y \le \frac{v}{c}\right) = F_Y\left(\frac{v}{c}\right) = 1 - e^{-\left(\frac{v}{cb}\right)^d}$, i.e., the cdf of Weibull(*cb, d*).

ii. $Y \sim \text{Weibull}(b,d) \Rightarrow P(Y \le b) = F_Y(b) = 1 - e^{-\left(\frac{b}{b}\right)^d} = 1 - e^{-1} = 0.632...$
Hence, b is the 63.2% quantile in any Weibull distribution.

iii. Let $V = \left(\frac{Y}{b}\right)^d$. Then $V > 0$ always, and, for $v > 0$, we have

$$(V \le v) \Leftrightarrow \left(Y \le b \cdot v^{\frac{1}{d}}\right) \Rightarrow P(V \le v) = F_Y\left(b \cdot v^{\frac{1}{d}}\right) = 1 - e^{-\left(\frac{bv^{1/d}}{b}\right)^d} = 1 - e^{-v},$$

i.e. the cdf of Exp(1) implying $V \sim \text{Exp}(1)$.

Solving for *Y*, we get $Y = bV^{\frac{1}{d}}$, where $V \sim \text{Exp}(1) = \Gamma(1,1)$ distribution.
Hence, from **problem 1A**, we have $E\left(V^r\right) = \Gamma(1+r)$ for all real $r > -1$, and

$$E(Y) = bE\left(V^{\frac{1}{d}}\right) = b\Gamma\left(1 + \frac{1}{d}\right)$$

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C. We return to the situation in the introduction and assume in addition that the common distribution of Y_i, $i = 1,2,\ldots,n$ is Weibull$(b,4)$, where we, for simplicity, have assumed the shape to be known, $d = 4$.

i. Show that the pdf of Y_i is

$$f_{Y_i}(y_i) = \begin{cases} \frac{4}{b^4} y_i^3 e^{-\left(\frac{y_i}{b}\right)^4} & \text{for } y_i > 0 \\ 0 & \text{for } y_i \le 0 \end{cases}$$

ii. Assuming $Y_1, Y_2, \ldots, Y_n$ iid as in the introduction, we want to perform a specification test of the additional Weibull$(b,4)$ assumption for the common distribution. In order to use a Pearson chi-square for this, we have grouped the 31 observations in **table 1** in 5 intervals resulting in the frequency table in **Table 2**.

Table 2. **Frequency table of grouped data from table 1**
(The upper limit in each interval is inclusive and the lower limit exclusive.)

Category j	Intervals	Observed frequency O_j	Expected frequency under H_0 E_j	$\frac{(O_j - E_j)^2}{E_j}$
1	0 - 24	5	7.409	0.783
2	24 - 30	11	7.657	1.460
3	30 - 36	8	8.122	0.002
4	36 - 42	4	5.394	0.360
5	> 42	3	2.387	0.157

Perform the Pearson chi-square test for H_0: $Y_i \sim$ Weibull$(b, 4)$, using approximately 5% as level of significance. Comment briefly on the result.

iii. It turns out that the maximum likelihood estimate (MLE) of b is $\hat{b}_{obs} = 33.2107$. Verify[2] that the expected frequency under H_0 of category 2 in **Table 2** is $E_2 = 7.657$ using this.

<<< Answer:

i. The pdf of Y, $f_{Y_i}(y_i) = 0$ for $y_i < 0$. For $y_i > 0$ we have

$$f_{Y_i}(y_i) = \frac{\partial}{\partial y_i}\left(1 - e^{-\left(\frac{y_i}{b}\right)^4}\right) = -e^{-\left(\frac{y_i}{b}\right)^4} \cdot \left(-\frac{4y_i^3}{b^4}\right) = \frac{4}{b^4} y_i^3 e^{-\left(\frac{y_i}{b}\right)^4}$$

ii. The Pearson test statistic is $Q = \sum_{j=1}^{5} \frac{(O_j - E_j)^2}{E_j} = 2.76...$ from **table 2.** Under H_0 Q is approximately χ^2_{df} - distributed with degrees of freedom, $df = 5 - 1 - 1 = 3$. The critical value is, therefore, the 0.95 quantile, $\chi^2_{3,\,0.95} = 7.81$ from the chi-square quantile table in Rice. We reject H_0 at approximately 5% level of significance if $Q > 7.81$. So, there is little evidence in the data for rejecting H_0.

iii. We have (with $\hat{b}_{obs} = 33.2107$),

$$E_2 = n \cdot \hat{P}(24 < Y_i \le 30) = n \cdot \left(F_{Y_i}(30) - F_{Y_i}(24)\right) = n \cdot \left(e^{-\left(\frac{24}{\hat{b}_{obs}}\right)^4} - e^{-\left(\frac{30}{\hat{b}_{obs}}\right)^4}\right) =$$

$$= 31 \cdot (0.761 - 0.514) = 7.657$$

>>>

[2] Small deviations in your answer may be due to rounding errors and therefore tolerated.

D. We now assume the following model to be true:

$Y_1, Y_2, \ldots, Y_n$ are iid with $Y_i \sim$ Weibull$(b, 4)$ for $i = 1, \ldots, n$.

i. Find the moment method estimator (MME) of *b*, and calculate the estimate.

[Hint. You may need the fact that $\Gamma\left(1+\frac{1}{4}\right) = 0.9064....$ **]**

ii. Show that the maximum likelihood estimator (MLE) of *b* is given by

$$\hat{b} = \left(\frac{1}{n}\sum_{i=1}^{n} Y_i^4\right)^{\frac{1}{4}}$$

iii. We assume that the general conditions for "good behavior" of MLE are fulfilled and that the asymptotic normal distribution for $\hat{b}$ is a satisfactory approximation.

The Stata output from maximum likelihood estimation of *b* is given in **Table 3.**

Table 3. Stata output from MLE estimation of *b*

```
     Maximum likelihood estimation

Log likelihood = -106.04338                     Number of obs     =         31

------------------------------------------------------------------------------
             |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
          /b |   33.2107    1.4912      22.27   0.000       30.288     36.133
```

- Describe briefly how the 95% CI in **Table 3** is calculated.
- Show how the CI in **table 3** can be transformed to an approximate 95% CI for $p = P(Y_i \le 30)$.

  [Hint. Explain first why $p = P(Y_i \le 30)$ is a strictly decreasing function of *b* in the present model. **]**

- Calculate the transformed CI and compare with the CI for *p* in **A.**

<<< **Answer:**

i. The model has one unknown parameter, *b*, so we need only one moment,

$\mu_1 = E(Y) = b\Gamma\left(1+\frac{1}{4}\right) = b \cdot 0.9064$ from section **C. iii**, estimated by the sample mean,

$\hat{\mu}_1 = \bar{Y}$.

The MME estimator is then $\tilde{b} = \frac{\hat{\mu}_1}{0.9064} = \frac{\bar{Y}}{0.9064}$ with estimate, $\tilde{b}_{obs} = \frac{30.82}{0.9064} = 34.00...$

ii. The log likelihood is (replacing y_i by the *i*-th observation)

$$l(b) = \sum_{i=1}^{n} \ln\ f_{Y_i}(y_i) = \sum_{i=1}^{n}\left(\ln 4 - 4\ln b + 3\ln y_i - b^{-4}y_i^4\right) =$$

$$= n\ln 4 - 4n\ln b + 3\sum_{i=1}^{n} y_i - b^{-4}\sum_{i=1}^{n} y_i^4$$

The derivative: $l'(b) = -\frac{4n}{b} + \frac{4}{b^5}\sum_{i=1}^{n} y_i^4 = 0$ for $b = \hat{b}_{obs}$, gives $\frac{1}{\hat{b}_{obs}^4}\sum_{i=1}^{n} y_i^4 = n \Leftrightarrow$

estimate: $\hat{b}_{obs} = \left(\sum_{i=1}^{n} y_i^4\right)^{\frac{1}{4}}$. The corresponding estimator is $\hat{b} = \left(\sum_{i=1}^{n} Y_i^4\right)^{\frac{1}{4}}$.

iii. The CI in **table 3** is calculated as $\hat{b}_{obs} \pm (1.96) \cdot \text{St.error}(\hat{b}) = 33.2107 \pm (1.96)(1.4912)$.
Transformation of the CI:

The parameter *p* is a function of *b* since $p = p(b) = P(Y_i \le 30) = 1 - e^{-\left(\frac{30}{b}\right)^4}$. It is a strictly decreasing function since $H(x) = 1 - e^{-x^4}$ is strictly increasing cdf (for Weibull(1,4)) with positive pdf ($H'(x)$) for all $x > 0$. Hence, if *b* increases, then $x = 30/b$ decreases and $H(30/b)$ decreases.

If $(L_b,\ U_b)$ denotes the approximately 95% CI for *b* (with $P\left(L_b \le b \le U_b\right) \approx 0.95$, we have, since $p(b)$ is decreasing, that $P\left(L_b \le b \le U_b\right) = P\left(p(U_b) \le p(b) \le p(L_b)\right) \approx 0.95$

Hence, the interval $\left(p(U_b),\ p(L_b)\right)$ is an approximately 95% CI for $p = P(Y_i \le 30)$.

Observed we get the CI for *p*:

$$\left(p(U_b),\ p(L_b)\right)_{obs} = \left(p(36.133),\ p(30.288)\right) = \left(1 - e^{-\left(\frac{30}{36.133}\right)},\ 1 - e^{-\left(\frac{30}{30.288}\right)}\right) =$$

$$= \left(0.378,\ 0,618\right)$$

This CI is shorter (length 0.240) than the CI for *p* in **A, (**$(0.340,\ 0.692)$ with length 0.352).

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