

ECON4140 Mathematics 3: postponed exam 2025-06-25

Problems / problem pages to be solved: 1 through 4, *including* this page.

All printed and written material may be used. A calculator is available in Inspira.

Grades given run from A (best) to E for passes, and F for fail.

- You are required to state reasons for all your answers.
- You are permitted to use any information stated in an earlier letter-enumerated item (e.g. “(a)”) to solve a later one (e.g. “(c)”), regardless of whether you managed to answer the former. A later item does not necessarily require answers from or information given in a previous one.

Suggested weighting (graders can deviate at their discretion): *Equal weight* across letter-enumerated parts («(a)», «(b)» etc.), Problem 4 counting as one.

Problem 1 For each real constant h , let

$$\mathbf{A} = \begin{pmatrix} h & 1 & 1 \\ h & 1 & h \\ 1 & 1 & h \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{M} = \begin{pmatrix} h & 1 & 1 & 1 \\ h & 1 & h & 1 \\ 1 & 1 & h & 1 \end{pmatrix} \quad (\text{so that } \mathbf{M} = (\mathbf{A} \mid \mathbf{b}))$$

You can use without proof that the determinant of $\mathbf{A}$ equals $-(h-1)^2$.

- (a) • Find the rank of $\mathbf{A}$, of $\mathbf{b}$ and of $\mathbf{M}$. (The answers might depend on h .)
- Show that the equation system $\mathbf{Ax} = \mathbf{b}$ always has a solution, and find the number of degrees of freedom. (The latter answer depends on h .)
- (b) For what h will $\mathbf{w} = (1, 0, 1)'$ be an eigenvector of $\mathbf{A}$?
- (c) $\mu = h - 1$ is an eigenvalue of $\mathbf{A}$. Find an associated eigenvector $\mathbf{u}$.
- (d) Consider, for each constant h , the quadratic form $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$. Show that:
- The form is never (i.e., not for any real h) negative semidefinite
 - There is at least one h such that the form is positive semidefinite (*hint*: an educated guess might be helpful, then verify that this h does the job).

Problem 2 Consider the difference equation (D) and the differential equation (E):

$$x_{t+2} + 6x_{t+1} + 9x_t = 144 \quad (\text{D})$$

$$\ddot{x}(t) + 6\dot{x}(t) + 9x(t) = 144 \quad (\text{E})$$

- (a)
 - Find the general solution of (D).
 - Find the general solution of (E).
- (b) Modify either (D) or (E) – *your choice!* – by replacing the right-hand side by « $\cos t$ ». *Explain how to find a particular solution of this modified equation.*
- (c)
 - Is (D) stable?
 - Is (E) stable?
 - Is it possible to decide stability of (D) and (E) without *solving* these equations?
- (d) Consider the differential equation system

$$\begin{aligned} \dot{x} &= x - 2y + Q \\ \dot{y} &= 8x - 7y - 2 \end{aligned} \quad (\text{S})$$

Find a constant Q such that $x(t)$ satisfies (E); *write out the calculations.*

- (e) Consider the *homogeneous differential equation system* associated with system (S). Sketch a phase diagram (with nullclines), and indicate some solution curves.

Problem 3 Let $x_0 > 0$. Consider the optimal control problem

$$\max_{u(t) \in [1,5]} \int_0^3 \left[x(t) - (u(t))^2 \right] dt \quad \text{subject to} \quad \dot{x} = 2ux \quad \text{where} \quad x(0) = x_0, \quad x(3) \text{ free}$$

It may be useful to note that since $x_0 > 0$, and u must be positive, $x(t)$ will necessarily be > 0 and *increasing* as function of t .

Let (x, u) be an admissible control that satisfies the conditions from the maximum principle, and assume that u is a continuous function of t .

- (a) • Write out the conditions from the maximum principle.
- If these conditions hold for (x, u) : can we then know that (x, u) will indeed be optimal, or do we need more calculations to tell?
- (b) Deduce from the maximum principle that $u(t)$ must be constant near the end – more precisely: that there is some $\tau < 3$ such that $u(t)$ is constant for $t \in (\tau, 3]$.

You are *not* asked to find τ !

- (c) Fact: there are cases where the optimal control is constant (call it $\bar{u}$) over the *entire* time interval $(0, 3)$. Suppose we are in such a situation:

Approximately how much would then the optimal value change, if the final time were reduced from 3 to 2.99?

You can express the answer in terms of $\bar{u}$ (and x_0) without calculating $\bar{u}$.

Problem 4 For each real constant b , consider – but do not try to solve – the nonlinear programming problem

$$\max \left(-x^2 - y^2 \right) \quad \text{subject to} \quad y + x^2 + 3e^{4x} \leq b \quad \text{and} \quad x \geq 0 \quad (\text{P})$$

Take for granted that the problem has a unique solution (x^*, y^*) ; that is, *one* admissible point maximizes, and no other admissible point is equally good.

- The question: Granted said fact about this problem, can we then know without solving – but possibly considering the functions involved – that the Kuhn–Tucker conditions *cannot hold anywhere else than* at (x^*, y^*) ?

(In a longer explanation: since we cannot solve by hand, we might hope that (say) a numerical search for points satisfying the Kuhn–Tucker conditions cannot possibly hit a sub-optimal point. The question asks if we can indeed rule out the existence of such.)

(end of problem set)