

ECON4140 Mathematics 3: on the 2025-06-25 exam

For general principles – including changes from 2024 – see the grading guideline for the ordinary (2025-05-28) exam. In what follows, the problems are restated in full. Then, boxed: any remarks, and how to solve.

Problem 1 For each real constant h , let

$$\mathbf{A} = \begin{pmatrix} h & 1 & 1 \\ h & 1 & h \\ 1 & 1 & h \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{M} = \begin{pmatrix} h & 1 & 1 & 1 \\ h & 1 & h & 1 \\ 1 & 1 & h & 1 \end{pmatrix} \quad (\text{so that } \mathbf{M} = (\mathbf{A} \mid \mathbf{b}))$$

You can use without proof that the determinant of $\mathbf{A}$ equals $-(h-1)^2$.

- (a)
 - Find the rank of $\mathbf{A}$, of $\mathbf{b}$ and of $\mathbf{M}$. (The answers might depend on h .)
 - Show that the equation system $\mathbf{Ax} = \mathbf{b}$ always has a solution, and find the number of degrees of freedom. (The latter answer depends on h .)
- (b) For what h will $\mathbf{w} = (1, 0, 1)'$ be an eigenvector of $\mathbf{A}$?
- (c) $\mu = h - 1$ is an eigenvalue of $\mathbf{A}$. Find an associated eigenvector $\mathbf{u}$.
- (d) Consider, for each constant h , the quadratic form $Q(\mathbf{x}) = \mathbf{x}'\mathbf{A}\mathbf{x}$. Show that:
 - The form is never (i.e., not for any real h) negative semidefinite
 - There is at least one h such that the form is positive semidefinite
(*hint*: an educated guess might be helpful, then verify that this h does the job).

Notes and remarks for Problem 1:

- (a) Note the pitfall that ranks are either 1 or 3, never 2.
- (b) Expect that once the y coordinate is zeroed out, an answer will not precisely point out any distinction between the necessity and the sufficiency of $h = -1/2$.
- (c) It is intentional to ask for «an» associated eigenvector, not them all.
- (d) A recurring pitfall is giving the quadratic form as a function in terms of a non-symmetric matrix. *Q always has a definiteness property* even when $\mathbf{A}$ is not symmetric. Hence it is *wrong* to claim that definiteness is ill-defined for $h \neq 1$; but it isn't worthless, as it shows familiarity with a related (but mis-used) condition.

How to solve Problem 1:

- (a) • $\mathbf{b}$ is a nonzero *vector* and thus has rank 1.
 $\mathbf{A}$ is square, and has full rank of 3 iff its determinant is nonzero, i.e. iff $h \neq 1$.
 When $h = 1$, $\mathbf{A}$ has three identical columns, and its rank reduces to that of $\mathbf{b}$, namely 1.
 $\mathbf{M}$ has its fourth column equal to its second. Deleting the fourth column, we are left with $\mathbf{A}$ and thus rank of 3 except: rank of 1 iff $h = 1$.
 • Solution exists because $\mathbf{A}$ and $\mathbf{M}$ have the same rank. Solution is unique except for $h = 1$. Then the common rank is 1, determining one variable, leaving *two* degrees of freedom.

(b) Multiply: $\mathbf{A}\mathbf{w} = \begin{pmatrix} h+1+1 \\ h+1+h \\ 1+1+h \end{pmatrix} = \begin{pmatrix} h+2 \\ 1+2h \\ h+2 \end{pmatrix}$ and so for the second element, we

must have $h = -1/2$. With $h = -1/2$, we find $\frac{3}{2} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, OK!

(c) $\mathbf{A} - \mu\mathbf{I} = \begin{pmatrix} h+1-h & 1 & 1 \\ h & 1+1-h & h \\ 1 & 1 & h+1-h \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ h & 2-h & h \\ 1 & 1 & 1 \end{pmatrix}$. Third row

equals first, solve $\begin{pmatrix} 1 & 1 & 1 \\ h & 2-h & h \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0}$; subtract h of the first row from the

second to get the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2-2h & 0 \end{pmatrix}$. Put $y = 0$ (that turns out to work even if $h = 1$) and then $z = -x$. An eigenvector is $(1, 0, -1)'$.

- (d) • The middle «1» yields $Q(0, 1, 0) = 1$ disproving negative semidefiniteness.
 • From (a) we saw that $h = 1$ yields a symmetric matrix of ones, where all principal minors are zero (hence ≥ 0) except the 1×1 , which are 1 hence ≥ 0 too. That gives positive semidefiniteness.

Problem 2 Consider the difference equation (D) and the differential equation (E):

$$x_{t+2} + 6x_{n+1} + 9x_n = 144 \quad (\text{D})$$

$$\ddot{x}(t) + 6\dot{x}(t) + 9x(t) = 144 \quad (\text{E})$$

- (a) • Find the general solution of (D).
• Find the general solution of (E).
- (b) Modify either (D) or (E) – *your choice!* – by replacing the right-hand side by « $\cos t$ ». *Explain how to find a particular solution of this modified equation.*
- (c) • Is (D) stable?
• Is (E) stable?
• Is it possible to decide stability of (D) and (E) without *solving* these equations?
- (d) Consider the differential equation system

$$\begin{aligned} \dot{x} &= x - 2y + Q \\ \dot{y} &= 8x - 7y - 2 \end{aligned} \quad (\text{S})$$

Find a constant Q such that $x(t)$ satisfies (E); *write out the calculations.*

- (e) Consider the *homogeneous differential equation system* associated with system (S). Sketch a phase diagram (with nullclines), and indicate some solution curves.

Notes and remarks for Problem 2: Every now and then there are those who pick the same recipe for difference equations and differential equations, without taking note that the numbers go different places. (That happened on the ordinary exam.) Hence asking both. Same goes for stability criteria. (D) and (E) have the double root «feature», that is a possible complication.

For (b): the course has not touched upon the formulae for sin or cos of *a sum*. But they are allowed to know, and to argue that they prefer that one.

(c) Given what we had time to cover this year, a negative answer to *asymptotic* stability is sufficient for (D).

(d) is a recurring problem type, with the requirement to write out the calculations rather than just using the book's formula.

(e): Phase diagrams are not the best answered part. Here there is a linear system, but the double root property has some peculiar visuals. Also the question also requires knowledge of what a homogeneous system is.

Nullclines are expected, and «reasonably horizontal/vertical» slopes across. (Nullclines and a couple of extra horizontal arrows were manually added to the diagram next page, hence the odd typography.)

How to solve Problem 2: We are anyway going to need the characteristic roots, solving $r^2 + 6r + 9$, that is a double root $r = -3$.

- (a) • Trying a constant particular solution, call it D . Then the left-hand side is $(1 + 6 + 9)D$ to match 144, i.e. $D = 9$. We find for the general solution:

$$(-3)^t \cdot [A + Bt] + 9$$

- Trying again a (different) constant particular solution, call it E . We get $9E = 144$, hence $E = 16$. We find for the general solution:

$$e^{-3t} \cdot [A + Bt] + 16$$

- (b) The following answer is even overkill for full score: «In (E), try $K \cos t + L \sin t$, insert in the left-hand side and fit K and L to $1 \cos t + 0 \sin t$.»

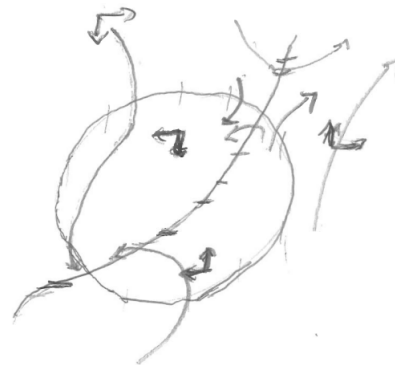
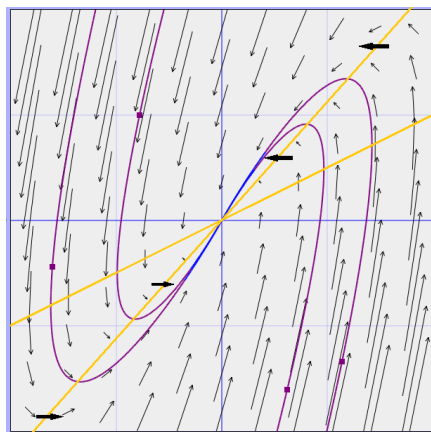
- (c) • The book yields two criteria for *asymptotic stability*, both acceptable for answers. The other is in terms of the coefficients a, b , $|a| < 1 + b < 2$, which fails as $9 > 1$, negating asymptotic stability (that is sufficient given the time we had in this course). The other is in terms of complex modulus; $|-3| > 1$ with same implication.

- Again the book yields two criteria for asymptotic stability. One is in terms of positive coefficients $6 > 0$ and $9 > 0$, hence asymptotically stable. The other is in terms of negative real parts, which is affirmed for -3 .

- Yes, by either argument just given.

- (d) $\ddot{x} = \dot{x} - 2\dot{y} = \dot{x} - 2[8x - 7y - 2] = 4 + \dot{x} - 16x + 7 \cdot 2y$. Insert $2y = x + Q - \dot{x}$ to get $4 + \dot{x} - 16x + 7 \cdot [x + Q - \dot{x}] = -6\dot{x} - 9x + 7Q + 4$. We are done if $7Q + 4 = 144$ i.e., $Q = 20$.

- (e) Left: a phase plot of the homogeneous system – too detailed to be asked from the exam papers, thus also included to the right: From the grading guideline to the *regular* 2025 exam – that level of detail and accuracy should be enough.



Problem 3 Let $x_0 > 0$. Consider the optimal control problem

$$\max_{u(t) \in [1,5]} \int_0^3 [x(t) - (u(t))^2] dt \quad \text{subject to} \quad \dot{x} = 2ux \quad \text{where} \quad x(0) = x_0, \quad x(3) \text{ free}$$

It may be useful to note that since $x_0 > 0$, and u must be positive, $x(t)$ will necessarily be > 0 and *increasing* as function of t .

Let (x, u) be an admissible control that satisfies the conditions from the maximum principle, and assume that u is a continuous function of t .

- (a) • Write out the conditions from the maximum principle.
 • If these conditions hold for (x, u) : can we then know that (x, u) will indeed be optimal, or do we need more calculations to tell?
- (b) Deduce from the maximum principle that $u(t)$ must be constant near the end – more precisely: that there is some $\tau < 3$ such that $u(t)$ is constant for $t \in (\tau, 3]$.

You are *not* asked to find τ !

- (c) Fact: there are cases where the optimal control is constant (call it $\bar{u}$) over the *entire* time interval $(0, 3)$. Suppose we are in such a situation:

Approximately how much would then the optimal value change, if the final time were reduced from 3 to 2.99?

You can express the answer in terms of $\bar{u}$ (and x_0) without calculating $\bar{u}$.

Remarks next page. This page: proposed solution for Problem 3. We need the Hamiltonian $H(x, u, p) = x - u^2 + 2pux$.

- (a) • Conditions:
- u : maximizes $x - u^2 + 2pux$ over $[1, 5]$
 - p : satisfies $\dot{p} = -1 - 2pu$ with $p(3) = 0$.
- [Include or omit: $\dot{x} = 2ux$ with $x(0) = x_0$.]
- No, nothing says we are going to get concavity for sufficient conditions.
- (b) When $t \approx 3$, then $p(t) \approx 0$ by the transversality condition. Then $\frac{\partial H}{\partial u} \approx -2u$ is < 0 for all $u \in [1, 5]$, and we will choose $u = 1$.
- (c) The answer $-0.01 \times [H \text{ inserted optimum and } t = 3]$. Then to identify the quantities involved: As $p(3) = 0$, the Hamiltonian reduces to $x(3) - \bar{u}^2$. The harder part is to calculate $x(3)$. Because $\dot{x} = 2x\bar{u}$ over the entire time horizon, $x(t) = x_0 e^{2\bar{u}t}$. So the answer is: $0.01 \times [\bar{u}^2 - x_0 e^{6\bar{u}}]$.

Some remarks: There is no « t » in the Hamiltonian, so one can write « $H(t, x, u, p)$ » or « $H(x, u, p)$ ». Also it is perfectly fine to use « λ » (current-value with no discounting).

- (a)
 - *Fine:* writing that the optimal control maximizes $2pux - u^2$ over $[1, 5]$;
Wrong: to claim stationarity wrt. u , missing endpoint solutions.
 - Elaboration is too much to ask. (Say: whenever we have interior solution $u = px$, nonlinear terms will be $-(px)^2 + 2(px)^2 \implies$ Arrow fails.)
- (b) The level of rigor of pointing out that x is bounded, to ensure $px \rightarrow 0$, is not at all needed for full score.
- (c) The significant bulk of the score should be handed out for pointing out that the answer is $-0.01 \times [H \text{ at optimum and final time}]$ – at least if there is an indication of understanding what it means. Allowing « $\bar{u}$ » as is without putting $\bar{u} = 1$, was intended the (not too likely) case of solving (c) without (b).

Problem 4 For each real constant b , consider – but do not try to solve – the nonlinear programming problem

$$\max \left(-x^2 - y^2 \right) \quad \text{subject to} \quad y + x^2 + 3e^{4x} \leq b \quad \text{and} \quad x \geq 0 \quad (\text{P})$$

Take for granted that the problem has a unique solution (x^*, y^*) ; that is, *one* admissible point maximizes, and no other admissible point is equally good.

- The question: Granted said fact about this problem, can we then know without solving – but possibly considering the functions involved – that the Kuhn–Tucker conditions *cannot hold anywhere else than at (x^*, y^*)* ?

(In a longer explanation: since we cannot solve by hand, we might hope that (say) a numerical search for points satisfying the Kuhn–Tucker conditions cannot possibly hit a sub-optimal point. The question asks if we can indeed rule out the existence of such.)

How to solve Problem 4: The following holds for full score:

«Concave program ($-x^2 - y^2$ concave, $y + x^2 + 3e^{4x}$ convex, x linear),
so the Kuhn–Tucker conditions are also sufficient.»

(Sure, possible to elaborate on how sufficiency grants that no suboptimal point can satisfy the conditions – but presumably, anything like the above would be written by someone who knows the theory well enough to catch what was intended. Thus I suggest to bear over with claims that the function x is convex – even if the key property is convexity of $-x$.)