

ECON4140 Mathematics 3: on the 2025-05-28 exam

- *Standard disclaimer:*
 - This note is not suited as a complete solution or as a template for an exam paper. It was written as guidance for the grading process – however, with additional notes and remarks for using the document in teaching later.
 - The document reflects what was expected in that particular semester, and which may not be applicable to future semesters.
- *Weighting is a suggestion*, and the graders can deviate.
- *Grade intervals:*

- *The «nominal» default percent score to grade conversion*, is given in the first line of the following table. The second line are [the Mathematics department's defaults, based on recommendations from Norsk matematikkråd](#):

	F (fail)	E	D	C	B	A
ECON4140	0 to 39	40 to 44	45 to 54	55 to 74	75 to 90	91 to 100
Math dept.	0 to 39	40 to 45	46 to 57	58 to 76	77 to 91	92 to 100

- *Practice from 2016 and forward* has often become an «A» threshold laxed somewhat, to distinguish the clear best-in-show paper(s) from other good ones. Graders should exercise their best judgement*, and in particular in light of this year's changes:
- *New in 2025 (I):* 4 hour exam rather than 3. Still open-book.
 - It has been communicated that the main reason is a general preference for a 4h format, but also a hope that the treatment of certain time-consuming problem types could improve. Phase diagrams have been mentioned with the disclaimer that it is neither a promise nor a threat.

However, facilitating time-consuming problem types would depend on the problem set not being overall too long – and workload *is* supposed to increase although not by a full one-third.
- *New in 2025 (II):* Syllabus changes, following up on changes in Mathematics 2.
 - Removed: Dynamic programming, proof by induction and double integrals. Hopefully, that compensates for the additional workload to learn the next items:
 - «New»: the basics of complex numbers. For ages it has been taught that «all real parts negative $\Rightarrow$ asymptotically stable differential equation (system)» – but it being voluntary to use phrases like «real parts» on an exam. Now it is outright syllabus to know what a real part is, and that makes possible questions like 3(a) – which is intended to be an easy one.

*For this exam: Maybe the course is still a mouthful from the background of (the revised) Mathematics 2; thus also the «B» requirement was relaxed, and some benefit of doubt applied. Suggestion in case of appeals: ask the administration for the «C» papers as benchmark.

- Mathematics 2 no longer has Kuhn–Tucker. The topic of static optimization is thus expected to demand higher workload on Mathematics 3 students. Problem 2(b) could have been assigned in Mathematics 2 up to 2023, but not in the 2024 version, which was presumed prior knowledge.
- Mathematics 2 no longer has integration by substitution nor separable differential equations. Might lead to increased workload *initially* to understand the basics, but once the Math 3 tools are learned, previous exam questions should not be any harder.
- On the other hand, Mathematics 2 now has first-order linear difference equations. Maybe that eases the learning curve for second-order, but likely not by much. Expect no more than a small impact on this item.

The problems, the solutions and remarks: In what follows, the problems are restated with general remarks, then how to solve. Problem 2 takes a page and a half, hence covered first:

Problem 2 Let $f(x, y) = x^3 e^y$ and let $g(x, y) = y + e^{x-3} - 2$.

- (a) For each of the functions f, g : Decide whether it is quasiconcave or quasiconvex or neither, on the (convex!) set of points where $x \geq 0$.

You can take for granted that «both quasiconcave and quasiconvex» is wrong answer; also you can take for granted that if you have shown one property for the set where $x > 0$, then by continuity it holds even when x is permitted to be zero.

Consider from now on the problem

$$\max f(x, y) \quad \text{subject to} \quad x \geq 0 \quad \text{and} \quad g(x, y) \leq 0 \quad (\text{P})$$

- (b) • Show that the point $(x_0, y_0) = (3, 1)$ satisfies the Kuhn–Tucker conditions associated to problem (P).
 • Find another (distinct from (x_0, y_0) , and admissible) point (x_1, y_1) satisfying the conditions.
- (c) Do we have *sufficient conditions* that verify that any of the points (x_0, y_0) and/or (x_1, y_1) is/are optimal? If so, which one(s) of those points?

Remarks first: Part (b) could up to 2024 have been Mathematics 2.

(a) is not unlike exam 2019 problem 4, assigned for a seminar (and re-assigned as in: if you didn't do it, make sure you do it now) – but experience is that answers could be a mixed bag, it isn't necessarily easy just because of being a «part (a)». Also it is there in order to give a hint to part (c), which does test a very specific piece of theory.

For part (a), there is no need to point out where that hint comes in – it suffices to show quasiconcavity of f on the set $\{(x, y); x > 0\}$, like done in the upcoming suggested solution. Similar will also go if one chooses to use the bordered Hessian (which vanishes when $x = 0$.)

How to solve Problem 2:

(a) g is the sum of convex functions and hence convex and hence quasiconvex.

$\ln f(x, y) = 3 \ln x + y$ is concave (qua sum of concaves) – thus f is quasiconcave.

(b) Let $L(x, y) = f(x, y) + \lambda x - \mu \cdot (y + e^{x-3} - 2)$. The Kuhn–Tucker conditions for an admissible point to solve, read:

$$3x^2e^y + \lambda = \mu e^{x-3} \quad (1)$$

$$x^3e^y = \mu \quad (2)$$

$$\lambda \geq 0 \quad (\text{and: } \lambda = 0 \text{ if } x > 0) \quad (3)$$

$$\mu \geq 0 \quad (\text{and: } \mu = 0 \text{ if } y + e^{x-3} < 2) \quad (4)$$

- (x_0, y_0) : $g(3, 1) = 1 + e^{3-3} - 2 = 0$ (equality!), and $x_0 = 3 > 0$ so $\lambda = 0$. The other conditions simplify to (1'): $27e = \mu e^0$; (2'): $27e = \mu$; (4'): $\mu \geq 0$; all hold with $\mu = 27e$ (which ≥ 0) and $\lambda = 0$.
- f has stationary points whenever $x = 0$. With $x = 0 = \mu = \lambda$, we only need admissibility. One example is the origin, but any point $(0, y_0)$ with $y_0 \leq 2 - e^{-3}$ will do.

(c) From part (a) we have a quasiconcave program. Sufficient conditions apply at (x_0, y_0) *because that is not a stationary point for f* .

That latter condition fails when $x = 0$, because $f'_x(0, y) = 3 \cdot 0^2 e^y$ and $f'_y(0, y) = 0^3 e^y$ both vanish.

Problem 1 Let $\mathbf{A} = \begin{pmatrix} 3 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 1 \end{pmatrix}$ and $\mathbf{W} = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 0 & 2 \\ 0 & 3 & 6 \\ 0 & 2 & 4 \end{pmatrix}$.

- (a) Find the rank of $\mathbf{W}$.
- (b) Show that each of the columns $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ of $\mathbf{W}$, is an eigenvector of $\mathbf{A}$, and all with the same (real!) eigenvalue.
- (c) $\mu = -1$ is an eigenvalue of $\mathbf{A}$. Find an associated eigenvector.
- (d) Do parts (a)–(c) provide enough information to decide the definiteness of the quadratic form $Q(\mathbf{z}) = \mathbf{z}'\mathbf{A}\mathbf{z}$? (Caution: be precise!)

(A quite standard problem – compare (b)(c)(d) to exam 2022#1 assigned for seminar 10, and to 2019#1(a)(b)(c) for seminar 5.) **How to solve:**

- (a) Elementary *column* operations (two of columns 1 and 2 subtracted from the third) would yield the answer faster, but in «Math 2 type row operations»: scale the first row by $1/2$ and subtract it from the second row; scale the third row by $1/3$ and subtract two of it from the fourth. Those operations do not

change the rank, which is then the same as of $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, i.e. the same as of

$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$, and that is $\underline{2}$, as the rightmost square block $= \mathbf{I}$ has full rank.

- (b) HOW TO: Multiply out all three: $\mathbf{A}\mathbf{w}_i$ to get $4\mathbf{w}_i$, confirming the claim with a common eigenvalue (which is $= 4$).

(Yes sure, $\mathbf{w}_3 = 2(\mathbf{w}_1 + \mathbf{w}_2)$ leading to $\mathbf{A}\mathbf{w}_3 = \mathbf{A}[2(\mathbf{w}_1 + \mathbf{w}_2)] = 2\mathbf{A}[\mathbf{w}_1 + \mathbf{w}_2] = 2 \cdot 4[\mathbf{w}_1 + \mathbf{w}_2] = 4\mathbf{w}_3$ once $\mathbf{A}\mathbf{w}_1$ and $\mathbf{A}\mathbf{w}_2$ are done – likely not spotted until after the fact, if at all.)

- (c) $\mathbf{A} - \mu\mathbf{I} = \mathbf{A} + \mathbf{I} = \begin{pmatrix} 4 & 2 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 2 & 2 \end{pmatrix}$ has the last two rows proportional, so delete one

of them. Solving will lead to $x_1 = x_2 = 0$ and $x_3 = -x_4$. Thus an eigenvector is $(0, 0, 1, -1)'$.

(Note: Those who miss the sign and eliminate on $\mathbf{A} - \mathbf{I}$ will still get an eigenvector, and should get score for showing that part of the method.)

- (d) Yes indeed, because *opposite-sign eigenvalues* $\implies$ the quadratic form assumes both signs (that does not require symmetry). Thus, indefinite.

Problem 3

- (a) Let r be a complex number such that $r^2 + 4kr + 3k = 0$.

For each real constant k : Find the *real part* of r .

- (b) Solve the difference equation $x_{t+2} + 2x_{t+1} + \frac{3}{2}x_t = e^{\pi t}$

- (c) Consider in this part the differential equation system (L) given by

$$\dot{\mathbf{z}} = \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \mathbf{z} \quad (\text{L})$$

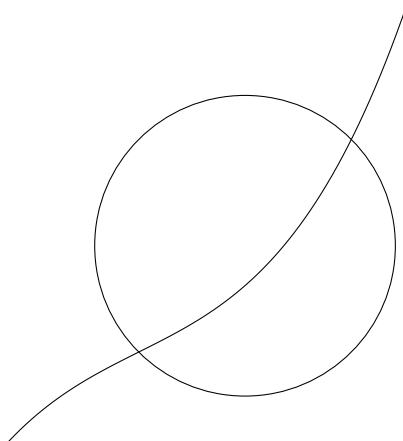
- Explain why the origin is a saddle point, and
- find the slope of the convergent (as $t \rightarrow +\infty$) non-constant solution paths.

Consider from now on the nonlinear differential equation system

$$\dot{x} = \frac{1}{2}(x-1)^2 + \frac{1}{2}(y-1)^2 - 1, \quad \dot{y} = \frac{1}{8}x^3 + \frac{1}{2}x - y \quad (\text{S})$$

The figure below gives a hint. Furthermore, you can expect some information from (a) to (c) to show up in the following.

- (d)
- Show that point $(2, 2)$ is a saddle point for the system (S).
 - There is *one* more equilibrium point. Find it and classify it.
(*Hint:* try integer coordinates – or sketch!)
- (e) Sketch a phase diagram for the system (S) and indicate some particular solution curves.
- If unable to do so, you can achieve up to half score on this part by instead sketching a phase diagram for system (L), and indicating some solution curves.*



Remarks first: Part (a): complex numbers is now syllabus in its own right.

It is deliberate that part (b) and the second equilibrium point of (S) in part (d) both involve an instance (namely $k = 1/2$) of that equation/root. Also, (d) first item essentially asks the same justification as (c) first item. And the phase diagram of (S) would locally around (2, 2) be approximated by that of (L); that does not mean that (L) is half the story of (S), but «up to half score» (graders can consider the fairness of that) did reflect the view that there is a job to do understanding what a phase diagram is.

Parts (b)(c) resemble 2016#2 (assigned for seminar 9), parts (c)(d) resemble 2019#1 (d)(e) (assigned for seminar 8; had a similar plot as hint) and (c)(d)(e) resemble 2017#1 (for seminar 9; also assigned: more phase diagrams).

How to solve Problem 3:

- (a) $r = -2k \pm \sqrt{4k^2 - 3k}$, and we need to distinguish between the situation where $k \cdot (4k - 3)$ is < 0 (that is, $k \in (0, 3/4)$) and the situation where $k \cdot (4k - 3)$ is ≥ 0 (that is, elsewhere):

- If $k \in (0, 3/4)$, the real part is $-2k$ (for both roots)
- If $k \notin (0, 3/4)$, the real part is r itself, namely $-2k \pm \sqrt{4k^2 - 3k}$

- (b) The characteristic equation is the instance $k = \frac{1}{2}$ of the equation in (a); $0 < 1/2 < 3/4$ so roots are non-real. Then we need « b » = the x_t coefficient = $3/2$ and « $\frac{-a}{2\sqrt{b}}$ » where a is the x_{t+1} coefficient, namely 2; thus, $\frac{-a}{2\sqrt{b}} = -\frac{2}{2\sqrt{3/2}} = -\sqrt{2/3}$ (or if you like, $= -\frac{1}{3}\sqrt{6}$).

Using the formula, we get for the homogeneous equation: $(3/2)^{t/2} [C \cos(\theta t) + D \sin(\theta t)]$ with θ such that $\cos(\theta) = \sqrt{2/3}$. For a particular solution, try $Ke^{\pi t}$:

$$K \left[e^{\pi t + 2\pi} + 2e^{\pi t + \pi} + \frac{3}{2}e^{\pi t} \right] = e^{\pi t} \quad \text{so:} \quad K = \frac{1}{e^{2\pi} + 2e^{\pi} + 3/2}$$

and so the general solution is:

$$(3/2)^{t/2} [C \cos(\theta t) + D \sin(\theta t)] + \frac{e^{\pi t}}{e^{2\pi} + 2e^{\pi} + 3/2} \quad \text{where } \cos(\theta) = -\sqrt{2/3}$$

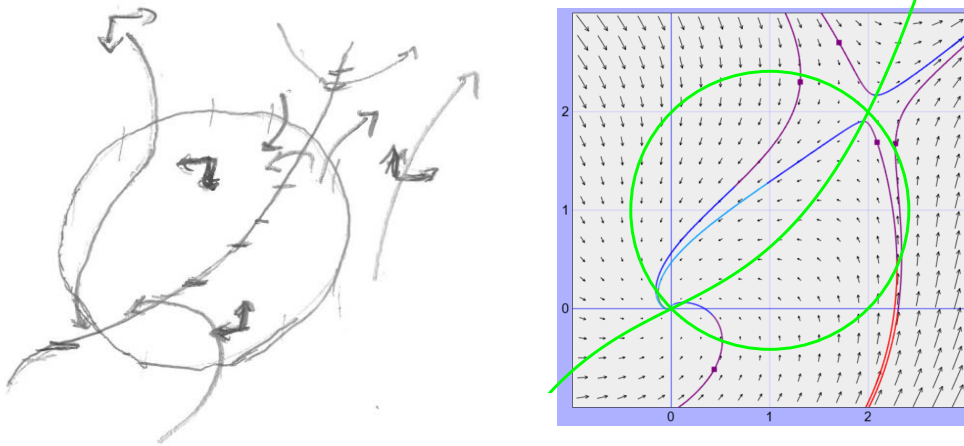
- (c)
- Saddle point, as the determinant of the coefficient matrix is < 0 .
 - We need the negative eigenvalue, call it γ . From that we find an eigenvector (v_1, v_2) such that $(1 - \gamma)v_1 + 1 \cdot v_2 = 0$; $\mathbf{v} = (1, -(1 - \gamma))'$ will do, and has a slope of $-(1 - \gamma)$. We need γ ; since the trace is zero, $\gamma = -\sqrt{-\det} = -\sqrt{3}$. The answer is $-(1 + \sqrt{3})$.

continued next page

(d) We need the Jacobian of the transformation:

$$\mathbf{J}_{x,y} = \begin{pmatrix} x-1 & y-1 \\ \frac{3}{8}x^2 + \frac{1}{2} & -1 \end{pmatrix}$$

- Let $x = y = 2$; then $\dot{x} = \frac{1}{2}1^2 + \frac{1}{2}1^2 - 1 = 0$ and $\dot{y} = \frac{1}{4} \cdot 2^3 + 2 - 4 = 2 + 2 - 4 = 0$. So an equilibrium point. Then $\mathbf{J}_{2,2} = \begin{pmatrix} 1 & 1 \\ 3/2 + 1/2 & -1 \end{pmatrix}$ is the same matrix as for (L), thus saddle.
 - The x -nullcline is a circle centered at $(1, 1)$ and with radius $\sqrt{2}$, and even the *disk* inside it won't cover too many points of integer coordinates; $(2, 2)$ is already taken, and if one of the coordinates is 1, then the other cannot be integer to end up on the circle. That leaves $(0, 0)$ to try; insert and verify it zeroes out both $\dot{x}$ and $\dot{y}$. $\mathbf{J}_{0,0} = \begin{pmatrix} -1 & -1 \\ 1/2 & -1 \end{pmatrix}$ has trace $= -2$ and determinant $= 3/2$, i.e. the same « $k = 1/2$ » instance of the (a) roots as we had in (b). Non-real roots with negative real part $-2k = -1$, it is asymptotically stable. (And spiralling inwards, but that need not be pointed out explicitly given the question.)
- (e) Left: a sketch that should be good enough for the graders to recognize that the crucial properties are captured: directions (including nullclines not-unlike those in the problem set) and solution curves in all regions. It is not required that the saddle path is one of the solution curves; it is decreasing at slope asymptotically $\rightarrow -(1 + \sqrt{3})$ like system (L). Right: a plot (with the nullclines in green).



(The «system (L)» phase plot would be a linearization around point $(2, 2)$.)

Problem 4 Let $T > 0$. Consider the optimal control problem

$$\max_{u(t) \in \mathbb{R}} \int_0^T \left((x+1) \cdot u - \frac{1}{2}(3x^2 + u^2) \right) dt \quad \text{when} \quad \dot{x} = u, \quad x(0) = x_0, \quad x(T) \geq 0$$

- (a) • Write out the conditions from the maximum principle.
 • Suppose an admissible pair (x, u) satisfies these conditions; do we then know that it will solve the problem?

Let (x, u) be an admissible pair satisfying the necessary conditions from the maximum principle, with adjoint (costate) p . You can take for granted that $u(t)$ is a continuous function of t .

- (b) • Deduce a differential equation system for (x, p) . (That is, with no « u » letter.)
 • Explain (in words, if you like) how the integration constants will be determined in each of the two cases $x(T) = 0$ and $x(T) \neq 0$.
 (c) Show that $u(T) > 0$ (i.e. at final time T).

Quite common problem type (e.g. 2022#3 for seminar 10.) How to solve:

We are going to need $H(x, u, p) = (x+1) \cdot u - \frac{1}{2}(3x^2 + u^2) + pu$.

- (a) First note that the conditions are sufficient, as H is concave in (x, u) : On $\mathbb{R}^2$,
 $H''_{xx}H''_{uu} - (H''_{xu})^2 = (-3) \cdot (-1) - 1^2 > 0$ with $H''_{uu} < 0$.

Then on to the conditions:

- The optimal u^* maximizes $(x+1) \cdot u - \frac{1}{2}(3x^2 + u^2) + pu$ over all real u ;
- p satisfies $\dot{p} = -u^* + 3x^*$ with $p(T) \geq 0$, and $p(T) = 0$ if $x(T) > 0$.

(In addition, the differential equation for x must hold, with the initial condition.)

- (b) The concave quadratic is maximized for the stationary point $u^* = x + p + 1$, and so $\dot{x} = x + 1 + p$. Then $\dot{p} = -(x + 1 + p) + 3x = 2x - p - 1$. That was the system.

One constant is determined by $x(0) = x_0$. The other is determined:

- By $x(T) = 0$, if that is the case;
- By $p(T) = 0$ otherwise.

For example, one can try to determine it by $x(T) = 0$ and then check if that leads to $p(T) \geq 0$; if not, then determine it by $p(T) = 0$ and check if that leads to $x(T) \geq 0$. (If that fails too, we have no solution.)

- (c) We have $u(T) = x(T) + p(T) + 1$, where $x(T) \geq 0$ (from the problem) and $p(T) \geq 0$ (from the transversality condition). So $u(T) \geq 1$.