

Complex numbers, ECON 4140

* Need to know for the exam?
Nothing! (But hopefully it helps.)

* For the lectures:

The phrase "real part" will be used a lot. Example:

$$\leadsto \ddot{x} + a\dot{x} + b = 0$$

$$r^2 + ar + b = 0$$

$$r_{1,2} = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

The real part of r is

r itself if r is real

$$-\frac{a}{2} \text{ if } b > \left(\frac{a}{2}\right)^2.$$

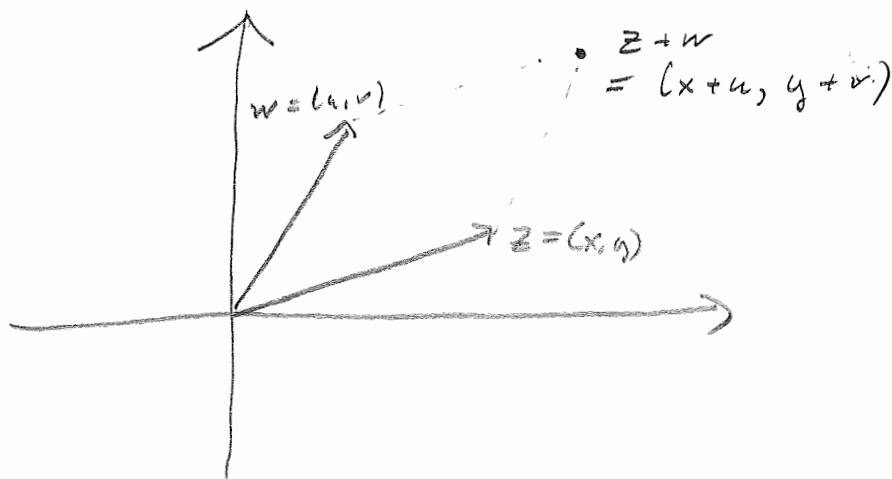
The real part in the exponent:

$$e^{t \cdot \operatorname{Re}(r)} \cdot \begin{cases} \text{const.}, & b < \left(\frac{a}{2}\right)^2 \\ \text{linear term}, & b = \left(\frac{a}{2}\right)^2 \\ \text{trigonometric}, & b > \left(\frac{a}{2}\right)^2 \end{cases}$$

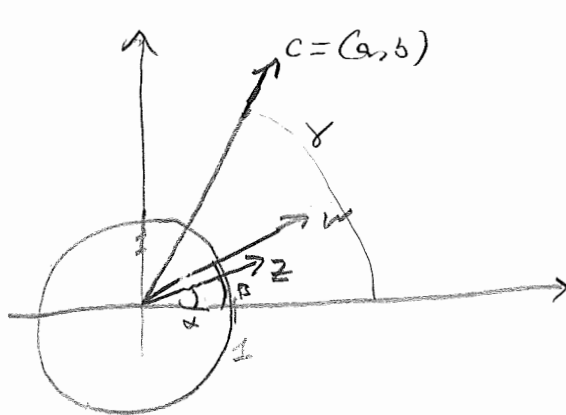
* Otherwise: A few properties - including exam relevant topics - just make more sense with complex numbers.

Geometry:

Consider pairs of reals (\mathbb{R}^2) ,
with coordinate-wise addition (like
if they were vectors):



Define multiplication as follows:



c has length = product of
the lengths
and angle = sum of
the angles

This is not a "dot product" type multiplication!

- Identify the real numbers with the
x-axis

Algebra:

In terms of pairs:

$$(x, y) + (u, v) = (x+u, y+v)$$

$$(x, y) \cdot (u, v) = (xu - yv, xv + yu)$$

Note:

$$(0, 1) \cdot (0, 1) = (0, -1) \cdot (0, -1)$$

$$= (-1, 0)$$

= the real number -1 .

So -1 has two square roots: $\pm i$,

where $i = (0, 1)$ (the "imaginary unit")

Just as a vector (x, y) can be written in terms of the basis vectors $\vec{e}_1 = (1, 0)$ and $\vec{e}_2 = (0, 1)$

as $x\vec{e}_1 + y\vec{e}_2$, we can write a complex number in terms of the real unit

$$1 = (1, 0) \quad \text{and} \quad i = (0, 1) \quad \text{as}$$

$$x1 + yi. \quad \text{We write } "x + iy" "$$

$x = \text{real part}$ $y = \text{imaginary part}$ (not "iy"!)

Multiplication rule: $i^2 = -1$.

$$\begin{aligned} (x + iy)(u + iv) &= xu + xiv + iyu + i^2yv \\ &= xu - yv + i(xv + yu) \end{aligned}$$

Modulus (absolute value!)

Real x : $|x| = \sqrt{x^2}$.

Complex $z = x + iy$: The Pythagorean

Theorem yields $|z| = \sqrt{x^2 + y^2}$

↑
single bars, not like
vector-norm!

Note: $|z| = \sqrt{(x+iy)(x-iy)}$.

The number $x - iy$ is called the
complex conjugate of $x + iy$,

and is usually denoted \bar{z} .

So overbar notation for vectors
is not recommended if you need
complex numbers!

Complex division $\frac{z}{w}$ is well-defined
when $w \neq 0$:

$$\frac{x+iy}{u+iv} \cdot \frac{(u-iv)}{(u-iv)} = \frac{z \bar{w}}{|w|^2}$$
$$= \frac{xu + yv + i(yu - xv)}{u^2 + v^2}$$

Complex numbers are numbers

$z = x + iy$ behaves "more like numbers than vectors":

- The product of two complex numbers is a complex number.

In contrast: the (dot) product of two vectors in \mathbb{R}^2 , is not a vector in \mathbb{R}^2 .

In particular: polynomials are just fine (not so for vectors.)

- We can divide. $\frac{z}{w}$ exists for $w \neq 0$. (Not so for vectors.)

And most rules carry over. E.g. $zw = 0$ implies that $z = 0$ or $w = 0$

- We can use complex numbers as elements in vectors and matrices.

→ Some modifications: If

$$z = x + iy \quad \text{and} \quad w = u + iv$$

$\begin{matrix} \uparrow & \uparrow \\ \mathbb{R} & \mathbb{R} \\ \text{real vectors} \end{matrix}$

we use $x \cdot u + y \cdot v + i(y \cdot u - x \cdot v)$ in place of the dot product

- Hermitian matrices, $a_{ij} = \overline{a_{ji}}$ in place of symmetric.

Useful facts:

$$z^n + C_{n-1}z^{n-1} + \dots + C_1z + C_0$$

$$= (z - z_1) \cdot (z - z_2) \cdot \dots \cdot (z - z_n)$$

holds for z_1, \dots, z_n , complex roots.

Hence $\det(A - \lambda \mathbf{I}) = (\lambda_1 - \lambda) \dots (\lambda_n - \lambda)$
has n complex eigenvalues, counted with multiplicity.

Ex.: $\begin{pmatrix} -1 & -1 \\ 4 & -1 \end{pmatrix}$ has eigenvalues $-1 \pm 2i$
and eigenvectors
 $\begin{pmatrix} 1 \\ 2i \end{pmatrix}, \begin{pmatrix} 1 \\ -2i \end{pmatrix}$.

$$\exp(iz) = 1 + iz - \frac{z^2}{2} - i\frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

$$\exp(-iz) = 1 - iz - \frac{z^2}{2} + i\frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

are conjugates, so their sum is real:

$$\frac{1}{2} [\exp(iz) + \exp(-iz)] = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots = \underline{\underline{\cos z}}$$

and the difference is purely imaginary:

$$\frac{1}{2} [\exp(iz) - \exp(-iz)] = i \left\{ z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right\} = i \sin z$$

Thus, $\exp(\pm iz) = \cos z \pm i \sin z$.

Modulus: $\cos^2 z + \sin^2 z = 1$ (Famously,
 $e^{i\pi} + 1 = 0$.)

non-real roots
of a quadratic
with real coeff's,
are a conjugate pair!

Applications I

* 2nd order linear homogeneous ODE, constant coeff's:

$$\ddot{x} + a\dot{x} + bx = 0$$

$$r^2 + ar + b = 0.$$

If no real roots, then two complex roots

$$\alpha \pm i\beta.$$

General solution $C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$$= e^{\alpha t} \left[C_1 \cos(\beta t) + C_1 i \sin(\beta t) + C_2 \cos(\beta t) - C_2 i \sin(\beta t) \right]$$

$$= e^{\alpha t} \left[P \cos(\beta t) + Q \sin(\beta t) \right]$$

$$P = C_1 + C_2, \quad Q = (C_1 - C_2)i.$$

Stability: $x \rightarrow 0$ for all C_1, C_2 as $t \rightarrow \infty$

\Downarrow
Both r_1 and r_2 have negative real part.

[Case: double root: $t e^{r t} \rightarrow 0 \Leftrightarrow r < 0.$]

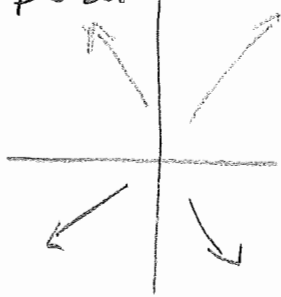
Applications II

* ODE systems.

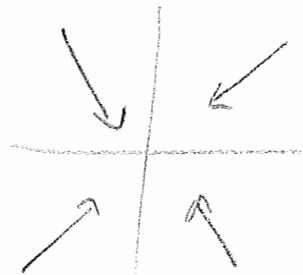
• Linear in \mathbb{R}^2 :

$$\dot{\bar{x}} = \bar{A} \bar{x}$$

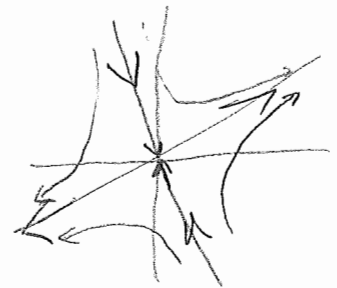
Both eigenvalues positive



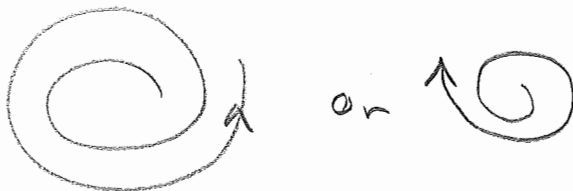
Both negative



$\lambda_2 > 0 > \lambda_1$



Non-real, $\text{Re } \lambda > 0$



Non-real, $\text{Re } \lambda < 0$

inwards spiral -

• Linear in \mathbb{R}^n : $x \rightarrow 0$ \forall constants of integration

$$\dot{\bar{x}} = \bar{A} \bar{x}$$

all eigenvalues have negative real part.

If \bar{A} has n linearly indep. eigenvectors (possibly complex!), then general solution

$$\bar{x}(t) = C_1 \bar{v}_1 e^{\lambda_1 t} + \dots + C_n \bar{v}_n e^{\lambda_n t}$$

• Nonlinear, $\dot{\bar{x}} = \bar{F}(\bar{x})$,

An equilibrium point \bar{x}^* is asymptotically stable

if all the eigenvalues of the Jacobian

$\frac{\partial \bar{F}}{\partial \bar{x}}(\bar{x}^*)$ have negative real part.

Applications III

Difference eq. systems

$$\bar{x}_{t+1} = \bar{A} \bar{x}_t, \quad \bar{x}_0 \text{ given.}$$

If \bar{x}_0 can be written as a linear combination of eigenvectors of \bar{A}

$$\bar{x}_0 = c_1 \bar{v}_1 + \dots + c_n \bar{v}_n$$

$$\begin{aligned} \text{then } \bar{x}_t &= c_1 \bar{A}^t \bar{v}_1 + \dots + c_n \bar{A}^t \bar{v}_n \\ &= c_1 \lambda_1^t \bar{v}_1 + \dots + c_n \lambda_n^t \bar{v}_n. \end{aligned}$$

This in particular holds true if \bar{A} is $n \times n$ with n lin. indep. eigenvectors.

$$\begin{aligned} \bar{x}_{t+1} = \bar{A} \bar{x}_t \text{ is stable } (\bar{x}_t \xrightarrow{t \rightarrow \infty} \bar{0} \neq \bar{x}_0) \\ \iff \\ \text{all } |\lambda_i| < 1 \\ \text{A} \\ \text{Complex modulus.} \end{aligned}$$