Concave programs.

$$v(\vec{b}) = \max f(\vec{x}) \quad \text{subject to} \quad g_j \le b_j,\ j = 1 \dots k$$
$$g_j = b_j \quad j = k+1, \dots m$$

where f concave, all g_j convex and all eq. constraints have <u>linear</u> g_j.

CQ? - Not exam relevant but if you see the term "Slater condition", that is it: there is an <u>admissible</u> $\vec{z}$ such that all <u>ineq</u> constraints are <u>inactive</u>.

We just assume Slater holds. Then we have cond's:

V has a supergradient $\vec{\lambda}$ at $\vec{b}$.

For any supergradient $\vec{\lambda}$:

* $\lambda_j \ge 0$ for the ineq. constraints, (=0 if inactive)
* If $\vec{x}^*$ solves the problem, then - with this $\vec{\lambda}$ - it will maximize
$$L(\vec{x}) = f(\vec{x}) - \vec{\lambda}'(\vec{g} - \vec{b}).$$

These cond's are also sufficient.

(But <u>existence</u> is not granted!)

→ Furthermore: <u>V is concave</u>.

Concave programs have concave <u>value functions</u>!

<u>indirect</u> utility

Recall also the shadow price interpretation

if $V(\vec{b}) = \max_{\vec{x} \in S} f(\vec{x})$, S given by

$$g_1(\vec{x}) - b_1 \begin{cases} \leq 0 \text{ or} \\ = 0 \end{cases}, \ldots, g_m(\vec{x}) - b_m \begin{cases} \leq 0 \text{ or} \\ = 0 \end{cases}$$

then [under suitable cond's], $\nabla V = \vec{\lambda}^T$.

Not to be stressed here: For $\nabla V = \vec{\lambda}^T$, the multipliers must exist, of course.

And if the CQ fails; example: let two constraints be $\vec{a}^T \vec{x} \leq 1$, $\vec{a}^T \vec{x} \leq b_2$.

$$\left.\frac{\partial V}{\partial b_2}\right|_{b_2 = 1} = ?$$

Furthermore: $\vec{\lambda}$ need not be continuous wrt $\vec{b}$.

Even if f and all the g are C^1, the value function $V(\vec{b})$ need not be!

Example: (b) Find a solution to

$\max(-\|\bar{x}\|)$ subject to

$\underbrace{(x_1-2)^2+\dots+(x_n-2)^2+e^{x_1-1}+\dots+e^{x_n-1}}_{\text{convex}} \le 2n$

concave

Note: $g(\bar{x}) = u(x_1)+\dots+u(x_n)$

when $u(z) = (z-2)^2+e^{z-1}$

so that $u'(z) = 2(z-2)+e^{z-1}$

$u'(1) = -1 < 0 < u'(2)$

so u has a global min $\tilde{z} \in (1,2)$

(and $u(1) = 2n$).

(a)

$L = -\|\bar{x}\| - \lambda\sum(u(x_i)-2)$

$\frac{\partial L}{\partial x_i} = -\frac{x_i}{\|\bar{x}\|} - \lambda u'(x_i)$ [except @ $\bar{x}=\bar{0}$, but that is not admissible!]

The symmetry in variables suggest $x_i = k$, all i,

i.e. $\bar{x} = k\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$, some k.

In that case:

$0 = \frac{\partial L}{\partial x_i} = -\frac{1}{\sqrt{n}} - \lambda u'(k)$: $\lambda > 0$ so $u(k)=2$ and $u'(k)<0$.

we see that $k=1$ works.

By suff. cond's, $\bar{x}^* = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$ solves.

(b) What happens to the optimal value when the RHS of the constraint is replaced by $z_n + \varepsilon$, $\varepsilon > 0$ small?

(c): Is the answer to Q1 an overestimate or an underestimate?

(b): $V' = \lambda = \frac{1}{\sqrt{n}}$.

(c): V is concave:

Linear approximation overestimates for $\varepsilon > 0$:

$$V(z_n) \le V(z_n + \varepsilon) \le V(z_n) + \lambda\varepsilon$$

always   concavity

Example 2: Consider the candidate point $k\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$

the problem

$$\max\left(-\|\vec{x}\|\sqrt{n}\right) \quad \text{s.t.} \quad w(x_1)+\dots+w(x_n) \le nq$$

$$w(z) = |z-1| + 3|z-2| + e^{z-1} - 4$$

q small negative, 0, or small positive.

Show: $\vec{x}^* = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$ for $q=0$.

$$L = -\|\vec{x}\|\sqrt{n} - \lambda\cdot\sum\left(w(x_i) - (4+q)\right)$$

$$\frac{\partial L}{\partial x_i} = -\frac{x_i}{\|\vec{x}\|}\sqrt{n} - \lambda\left(\operatorname{sgn}(x_i-1) + 3\operatorname{sgn}(x_i-2) + e^{x_i-1}\right)$$

note:
$$\begin{cases} w'(1^-) = -3 \\ w'(1^+) = -1 \\ w'(2^-) = e-2 \quad (>0) \\ w'(2^+) = e+4 \end{cases}$$

For $q>0$: can choose $k<1$, $x_i<1$

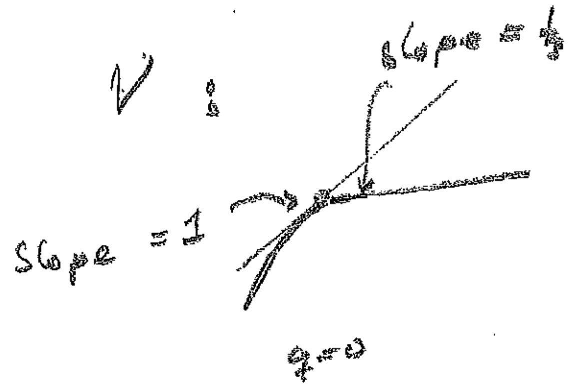
$$\frac{\partial L}{\partial x_i} = -1 + \lambda\left(4-e^{x_i-1}\right), \quad \lambda = \frac{1}{4-e^{k-1}}$$

For $q<0$: Must choose $k>1$, $x_i>1$

$$-1 + \lambda\left(2-e^{x_i-1}\right) \qquad \lambda = \frac{1}{2-e^{k-1}} > 1$$

So any $\lambda \in \left[\frac{1}{3}, 1\right]$ will do, and then $\vec{x}^* = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$ maximizes L.

V :



We know (suff cond's) that

- the suggested solution for $q < 0$ is optimal, with $\lambda = \dfrac{1}{2 - e^{z-1}}$

  So slope @ $q = 0^-$: $\dfrac{1}{2 - e^{1-1}} = 1$

- the suggested solution for $q > 0$ is optimal, with $\lambda = \dfrac{1}{4 - e^{z-1}}$

  So slope @ $q = 0^+$: $\dfrac{1}{4 - e^{0}} = \dfrac{1}{3}$

So any number $\in [\frac{1}{3}, 1]$ is a supergradient for V @ $q = 0$

Because $\bar{x}^* = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ maximizes L, when $\lambda \in [-\frac{1}{3}, 1]$, it solves the problem.

Quasoconcavity - based sufficient cond's, C^1 case

Consider a nonlinear programming problem with f quasoconcave. Then:

Suppose $\vec{x}^*$ satisfies the K-T cond's, with multipliers $\vec{\lambda}$, and such that all $\lambda_j g_j(\vec{x})$ are quasiconvex

If $\nabla f(\vec{x}^*) \neq 0$ (or $\vec{x}^*$ is an unconstrained max for f)

then x^* solves the maximization problem.

In particular, these conditions hold for quasoconcave programs:

$$\max \ f(\vec{x}) \quad \text{s.t.} \quad \begin{array}{ll} g_j(x) \leq b_j & j = 1 \ldots k \\ g_j(x) = b_j & j = k+1 \ldots m \end{array}$$

where f quasiconcave; all g_j quasiconvex, all eq. constraints have quasilinear g_j.

Ex: all utility functions quasiconcave, consider

$\max \ u_0(\vec{x})$ s.t. $u_j(x) \geq w_j$ and

full allocation

linear constraints on the allocation
[nonnegativity? market clearing? free disposal?]

<u>Proof</u> for the sufficiency cond'n: (case $\nabla f(\vec{x}^*) \neq 0$)

The proof has so much microeconomics that I cover it for the "utility functions" setup:

$$\max\ u_0(\vec{x}) \quad \text{subject to} \quad u_j(\vec{x}) \geq w_j \quad j = 1, \ldots, k$$

full allocation

$$\text{and} \quad h_j(\vec{x}) = c_j \quad j > k.$$

- Stationary Lagrangian when

$$\vec{0}^T = \nabla u_0(\vec{x}^*) + \sum_{j=1}^{k} \lambda_j \nabla u_j(\vec{x}^*) + \sum_{j>k} \mu_j \nabla h_j(\vec{x}^*).$$

$\mu_j = \lambda_j$

- Suppose for contradiction that $\vec{x}^*$ does <u>not</u> solve. Then there is some admissible $\vec{x}$ s.t. $u_0(\vec{x}) > u_0(\vec{x}^*)$

The proof tries to construct "something close to a Pareto improvement": $\vec{x}$ will improve #0, and also every #j for which $\lambda_j > 0$: for then, $u_j(\vec{x}^*) = w_j$ and that is <u>minimal</u>; so $u_j(\vec{x}) \geq u_j(\vec{x}^*)$. | And, $h_j(\vec{x}) = c_j = h_j(\vec{x}^*)$

By quasiconcavity, $\lambda_j \nabla u_j(x^*)(\vec{x} - x^*) \geq 0$

By quasilinearity hence quasiconcavity, $\mu_j \nabla h_j(\vec{x}^*)(\vec{x} - \vec{x}^*) \geq 0$

And for u_0, but we have more: since $\nabla u_0(\vec{x}^*) \neq 0$ we indeed have $\nabla u_0(\vec{x}^*)(\vec{x} - \vec{x}^*) > 0$ (strict!).

So from $\vec{0}^T = \frac{\partial L}{\partial x}(x^*)$ we get $0 = \frac{\partial L}{\partial x}(x^*)(\vec{x} - \vec{x}^*)$

$$= \underbrace{\nabla u_0(\vec{x}^*)(\vec{x} - \vec{x}^*)} + [\text{nonnegative terms}]$$

> 0, contradiction.