Complex numbers, ECON 4140

* Need to know for the exam:
<u>Nothing</u>! (But hopefully it helps.)

* For the lectures:
The phrase "real part" will be used a lot. Example:

$$\rightsquigarrow \quad \ddot{x} + a\dot{x} + b = 0$$

$$r^2 + ar + b = 0$$

$$r_{1,2} = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$$

The <u>real part</u> of r is
r itself if r is real
$-\frac{a}{2}$ if $b > (a/2)^2$.

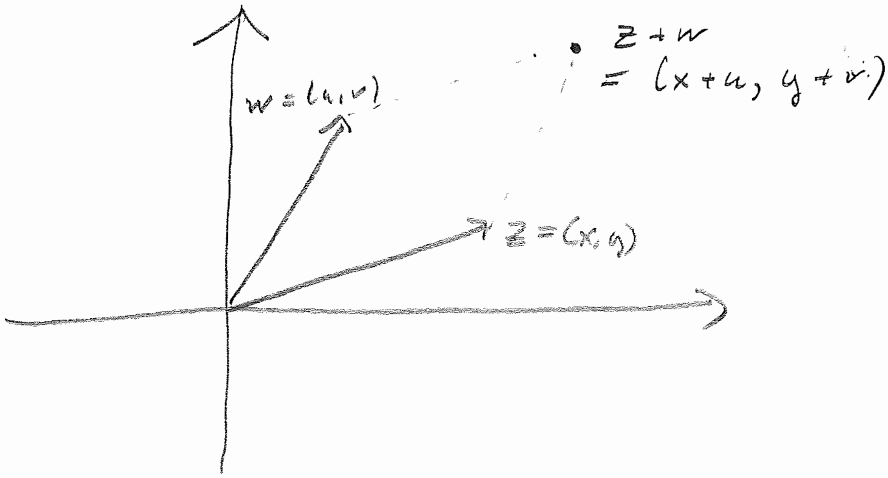
The real part in the exponent:

$$e^{t \cdot Re(r)} . \quad \begin{cases} \text{const.}, & b < (a/2)^2 \\ \text{linear term}, & b = (\ ) \\ \text{trigonometric}, & b > \left(\frac{a}{2}\right)^2 \end{cases}$$

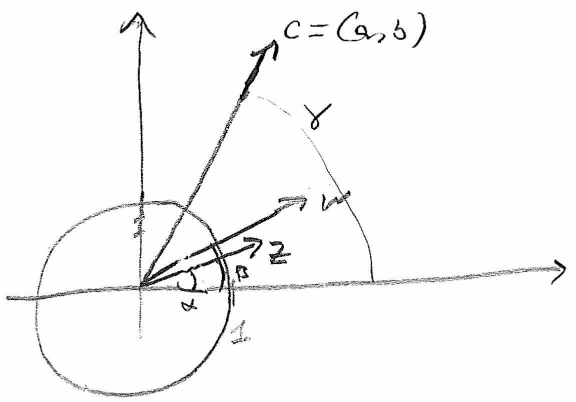
* Otherwise: A few properties – including exam relevant topics – just make more sense with complex numbers.

Geometry:

Consider pairs of reals ($\mathbb{R}^2$), with coordinate-wise addition (like if they were vectors):



Define multiplication as follows:



c has length = product of the lengths and angle = sum of the angles

This is not a "dot product" type multiplication!

- Identify the real numbers with the x-axis

Algebra:

In terms of <u>pairs</u>:

$$(x,y) + (u,v) = (x+u, y+v)$$

$$(x,y) \cdot (u,v) = (xu - yv, xv + yu)$$

Note:

$$(0,1) \cdot (0,1) = (0,-1) \cdot (0,-1)$$
$$= (-1, 0)$$
$$= \text{the real number } -1.$$

So -1 has two square roots: $\pm i$, where $i = (0,1)$ (the "imaginary unit")

Just as a vector (x,y) can be written in terms of the basis vectors $\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0,1)$

as $x\vec{e}_1 + y\vec{e}_2$, we can write a complex number in terms of the real unit $1 = (1,0)$ and $i = (0,1)$ as $x1 + yi$. We write " $x + iy$ "

x = real part

y (not "iy"!) = imaginary part

Multiplication rule: $i^2 = -1$.

$$(x+iy)(u+iv) = xu + xiv + iyu + i^2 yv$$
$$= xu - yv + i(xv + yu)$$

Modulus (absolute value!)

Real x: $|x| = \sqrt{x^2}$.

Complex $z = x+iy$: The Pythagorean theorem yields $|z| = \sqrt{x^2+y^2}$

single bars, not like vector norm!

Note: $|z| = \sqrt{(x+iy)(x-iy)}$.

The number $x-iy$ is called the complex conjugate of $x+iy$, and is usually denoted $\bar{z}$.

So overbar notation for vectors is not recommended if you need complex numbers!

Complex division

$\frac{z}{w}$ is well-defined when $w \neq 0$:

$$\frac{x+iy}{u+iv} \cdot \frac{(u-iv)}{(u-iv)} = \frac{z\bar{w}}{|w|^2}$$

$$= \frac{xu+yv+i(yu-xv)}{u^2+v^2}$$

Complex numbers are $\underline{\underline{\text{numbers}}}$

$z = x + iy$ behaves "more like numbers than vectors":

- The product of two complex numbers is a complex number.

  In contrast: the (dot) product of two vectors in $\mathbb{R}^2$, is not a vector in $\mathbb{R}^2$.

In particular: polynomials are just fine (not so for vectors.)

- We can divide. $\frac{z}{w}$ exists for $w \neq 0$.

  (Not so for vectors.)

And most rules carry over. E.g. $zw = 0$ implies that $z = 0$ or $w = 0$

- We can use complex numbers as elements in vectors and matrices.

  → Some modifications: If

  $$\mathbb{z} = \underbrace{\mathbb{x} + i\,\mathbb{y}}_{\text{real vectors}} \quad \text{and} \quad \mathbb{w} = \mathbb{u} + i\,\mathbb{v}$$

  we use $\mathbb{x}\cdot\mathbb{u} + \mathbb{y}\cdot\mathbb{v} + i(\mathbb{y}\cdot\mathbb{u} - \mathbb{x}\cdot\mathbb{v})$ in place of the dot product

  - Hermitean matrices, $a_{ij} = \bar{a}_{ji}$ in place of symmetric.

Useful facts:

$$z^n + c_{n-1} z^{n-1} + \dots + c_1 z + c_0$$
$$= (z - z_1) \cdot (z - z_2) \cdot \dots \cdot (z - z_n)$$

holds for $z_1, \dots z_n$, complex roots.

Hence $\det(A - \lambda I) = (\lambda_1 - \lambda) \dots (\lambda_n - \lambda)$

has n complex eigenvalues, counted with multiplicity.

Ex.: $\begin{pmatrix} -1 & -1 \\ 4 & -1 \end{pmatrix}$ has eigenvalues $-1 \pm 2i$

and eigenvectors

$\begin{pmatrix} 1 \\ 2i \end{pmatrix}$, $\begin{pmatrix} 1 \\ -2i \end{pmatrix}$.

non-real roots of a quadratic with <u>real</u> coeff's, are a <u>conjugate pair</u>!

$$\exp(iz) = 1 + iz - \frac{z^2}{2} - i\frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$
$$\exp(-iz) = 1 - iz - \frac{z^2}{2} + i\frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

are conjugates, so their <u>sum</u> is real

$$\frac{1}{2}\left[\exp(iz) + \exp(-iz)\right] = 1 - \frac{z^2}{2} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots = \underline{\cos z}$$

and the <u>difference</u> is purely imaginary:

$$\frac{1}{2}\left[\exp(iz) - \exp(-iz)\right] = i\left\{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right\} = i \sin z$$

Thus, $\exp(\pm iz) = \cos z \pm i \sin z$.

Modulus: $\cos^2 z + \sin^2 z = 1$. (Famously, $\underline{e^{i\pi} + 1 = 0}$.)

Applications I

* 2^{nd} order linear homogeneous ODE, constant coeff's:

$$\ddot{x} + a\dot{x} + bx = 0$$

$$r^2 + ar + b = 0.$$

If no real roots, then two complex roots $\alpha \pm i\beta$.

General solution $C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$$= e^{\alpha t}\left[C_1 \cos(\beta t) + C_1 i \sin(\beta t) + C_2 \cos(\beta t) - C_2 i \sin(\beta t)\right]$$

$$= e^{\alpha t}\left[P \cos(\beta t) + Q \cos \beta t\right]$$

$$P = C_1 + C_2, \quad Q = (C_1 - C_2) i.$$

<u>Stability</u>: $x \to 0$ for all C_1, C_2 as $t \to \infty$

$\Updownarrow$

Both r_1 and r_2 have negative real part.

[Case: double root: $te^{rt} \to 0 \Leftrightarrow r < 0$.]

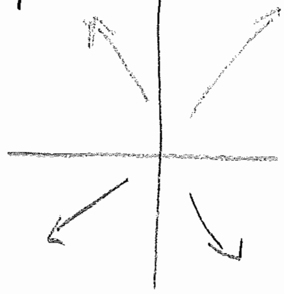
Applications II

* ODE systems.

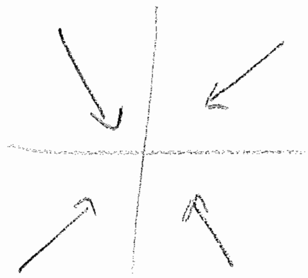
- Linear in $\mathbb{R}^2$:

$$\dot{\bar{x}} = \bar{A}\,\bar{x}$$

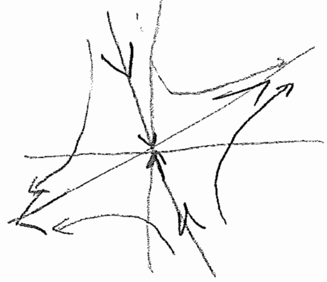
Both eigenvalues positive



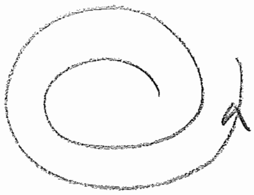
Both negative



$\lambda_2 > 0 > \lambda_1$



Non-real, Re $\lambda > 0$



or

Non-real, Re $\lambda < 0$

inwards spiral.

- Linear in $\mathbb{R}^n$: $\quad x \to 0 \quad \forall$ constants of integration

$$\Updownarrow$$

$\dot{\bar{x}} = \bar{A}\dot{x}$ $\qquad$ all eigenvalues have negative real part.

If $\bar{A}$ has n distinct eigenvectors (possibly complex!), then general solution

$$\bar{x}(t) = C_1 \bar{v}_1 e^{\lambda_1 t} + \dots + C_n \bar{v}_n e^{\lambda_n t}$$

- Nonlinear, $\dot{\bar{x}} = \bar{F}(\bar{x})$,

An equilibrium point $\bar{x}^*$ is asymptotically stable

if all the eigenvalues of the Jacobian $\dfrac{\partial F}{\partial \bar{x}}(\bar{x}^*)$ have negative real part.

Applications III

Difference eq. systems

$$\bar{x}_{t+1} = \bar{A} x_t \quad , \quad \bar{x}_0 \text{ given.}$$

If $\bar{x}_0$ can be written as a linear combination of eigenvectors of $\bar{A}$

$$\bar{x}_0 = c_1 \bar{v}_1 + \dots + c_n \bar{v}_n$$

then

$$\bar{x}_t = c_1 \bar{A}^t \bar{v}_1 + \dots + c_n \bar{A}^t \bar{v}_n$$

$$= c_1 \lambda_1^t \bar{v}_1 + \dots + c_n \lambda^t \bar{v}_n .$$

This in particular holds true if $\bar{A}$ is $n \times n$ with n lin. indep. eigenvectors.

$$\bar{x}_{t+1} = \bar{A}\,\bar{x}_t \text{ is stable } \left(\bar{x}_t \underset{t\to\infty}{\to} \bar{0} \quad \forall \bar{x}_0\right)$$

$$\Updownarrow$$

$$\text{all } |\lambda_i| < 1$$

complex modulus.