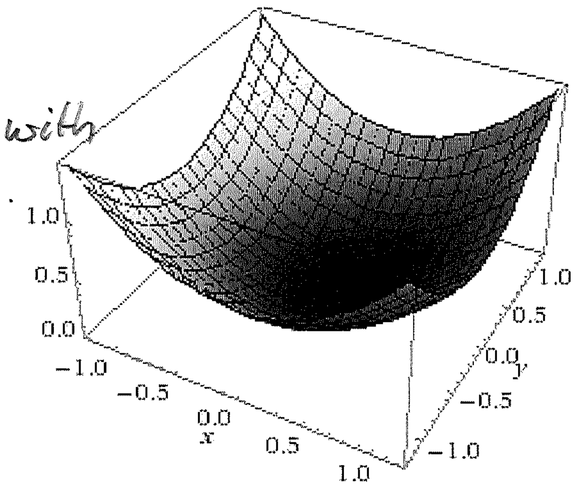
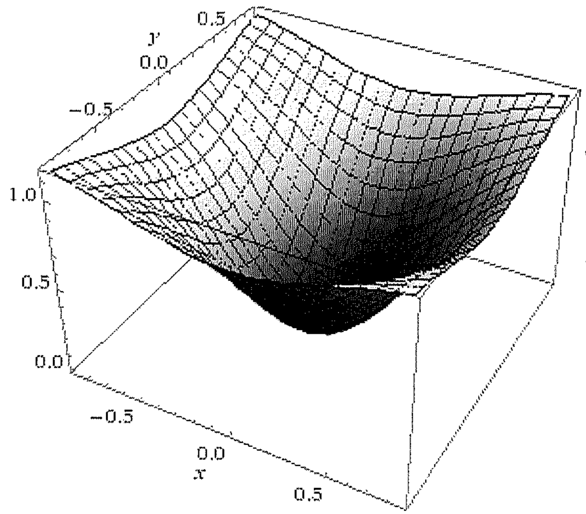
Convex

Fill up with blue ink. The blue set is convex

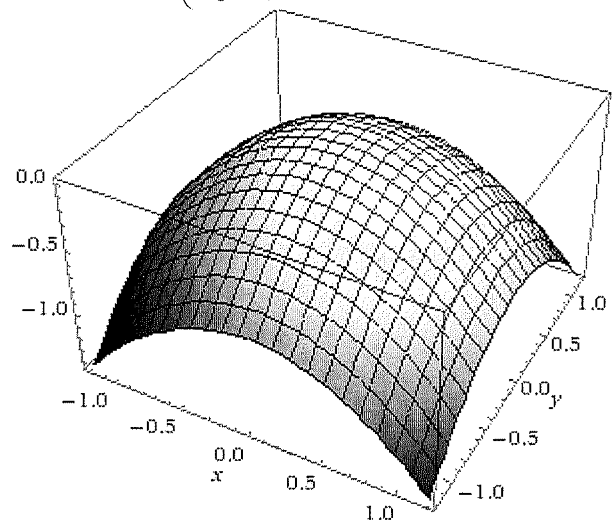


Quasiconvex

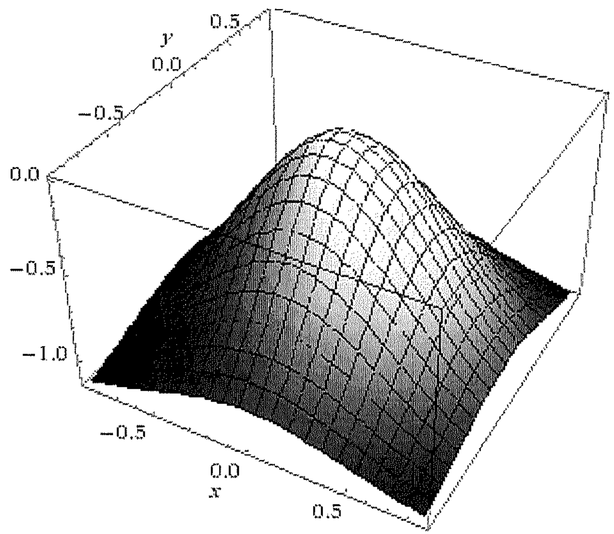
For whatever level we fill up with water: Put a layer of floating blue ink on top. That should be a convex set



Concave



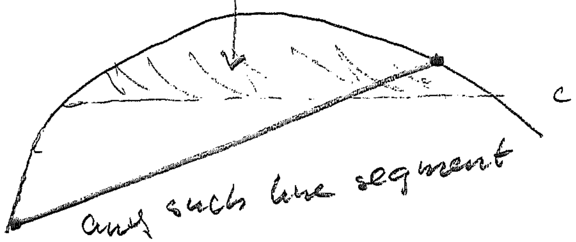
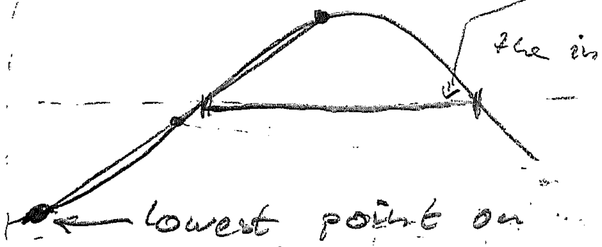
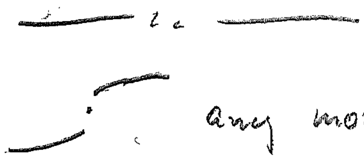
Quasiconcave



* A function that is both concave and convex, is linear (well, affine: it could have a constant term). Therefore, we call a function *quasilinear* if it is both quasiconcave and quasiconvex. *Example*: any strictly monotone transformation of a linear $\mathbf{a}^{\mathrm{T}}\mathbf{x}$.

* The 2019 version of Math 3 does not stress *strict* quasiconcavity/quasiconvexity. Though, it is possiblefor a function to be both. (*Example:* Any strictly monotone of a single variable!)

* The above quasiconcave is a transformation of a concave, but there are quasiconcaves that cannot be written as such a transformation.

	Concave	Quasiconcave
What set is convex?	$\{(\bar{x}, y);\ f(\bar{x}) \geq c\}$ for every level c.	$\{\bar{x};\ f(\bar{x}) \geq c\}$, every level c.
		
Graph never below:	any such line segment	lowest point on any such line segment.
$f(\lambda \bar{x}^{(1)} + (1-\lambda)\bar{x}^{(0)})$ is	$\geq \lambda f(\bar{x}^{(1)}) + (1-\lambda) f(\bar{x}^{(0)})$	$\geq \min\{f(\bar{x}^{(0)}), f(\bar{x}^{(1)})\}$
continuous?	On interior	Need not be 
f' ?	"Essentially so", on interior – but you cannot just find a stationary point. In $\mathbb{R}^1$, single variable: There is some g such that $f(b) - f(a) = \int_a^b g(x)\,dx$ (all a, b) and: g nonincreasing	any monotone ...
Example	Cobb-Douglas, Σ exponents ≤ 1	Cobb-Douglas, Σ exponents > 0

all exponents assumed > 0.

Property preserved?	Concave	Quasiconcave
Under positive scaling?	αf concave if $\alpha \geq 0$, f concave	Similar: αf quasiconcave if $\alpha \geq 0$, f —— " ——.
Under sums?	Yes, $f+g$ concave if f and g both concave.	No. Counterex: $\sqrt{\lvert x-2\rvert} + \sqrt{\lvert x+2\rvert}$.
Under composition $h(\vec{x}) = g(f(\vec{x}))$?	Need extra assumption: If f, g both concave & g nondecreasing then h is concave	Yes: If f, g both quasiconcave then h is quasiconcave.
Under min? $h(\vec{x}) = \min\{g(\vec{x}), f(\vec{x})\}$	Yes. If f, g concave, then h concave.	Yes. If f, g both quasiconcave, then h quasiconcave.

[No "max" – that is for convexity / quasiconvexity!]

C^1 condition: The first-order approximation for change from $\vec{x}^*$ to $\vec{x}$, $\nabla f(x^*)(x - x^*)$ is

$\geq$ true change $f(\vec{x}) - f(\vec{x}^*)$ iff f concave | ≥ 0 whenever $f(\vec{x}) > f(\vec{x}^*)$ iff f quasiconcave

C^2 cond. Hessian negative semidef. | Bordered Hessian condition.

Nonlinear programming properties

$\max \; f(\vec{x}) \quad \text{s.t} \quad \underset{j=\dots}{g_j(\vec{x}) \le b_j} \qquad \underset{k=\dots}{h(x_k) = c_k}$

Concave program

f concave, all g_j convex, all h_k linear

Quasiconcave program

f quasiconcave, all g_j quasoconvex
all h_j quasilinear

Sufficiency Suppose an admissible $\vec{x}^*$ satisfies the K-T cond's

Concave: $\vec{x}^*$ solves

Quasiconcave: If $\nabla f(\vec{x}^*) \neq \vec{0}^T$ then x^* solves

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K-T w/o differentiability assumptions,
concave case:

Value $= v(\vec{b}, \vec{c})$ concave.

$\Rightarrow$ has a supergradient $(\vec{\lambda}, \vec{\mu})$ at each point $(\vec{b}, \vec{c})$; for any such;

an optimal $\vec{x}^*$ maximizes the Lagrangian