Convex sets

(no: we do not speak about "concave" sets)

Recall: • Convex combination of $\vec{u}$ and $\vec{v}$
$= \lambda\vec{u} + (1-\lambda)\vec{v}$ where $\lambda \in [0,1]$

• convex combination of $\vec{v}^{(1)}, \ldots, \vec{v}^{(k)}$
$= \lambda_1 \vec{v}^{(1)} + \ldots + \lambda_k \vec{v}^{(k)}$, each $\lambda_i \geq 0$, $\sum_{i=1}^{k} \lambda_i = 1$.

Def: Let S be a subset of some vector space, say $\mathbb{R}^n$
(so that lin. comb's are well-defined!)

S is convex if, whenever $\vec{u} \in S$ and $\vec{v} \in S$,
we also have $\lambda\vec{u} + (1-\lambda)\vec{v} \in S$, any $\lambda \in (0,1)$.

Also: strictly convex if $\lambda\vec{u} + (1-\lambda)\vec{v}$ is in addition never a boundary point (if $\lambda \in (0,1)$) unless $\vec{u} = \vec{v}$ and $\vec{v}$ on boundary.

Ex: • $\mathbb{R}^n$ itself!
And $\emptyset$ and any singleton – but no other finite set!

• In $\mathbb{R}^1$: the intervals! Indeed, "convex set" is a generalization of "interval". And; book uses $[\vec{u}, \vec{v}]$ for $\{\vec{w};$ conv. comb of $\vec{u}, \vec{v}\}$.

• Given $\vec{a}, r$: $\{\vec{x};\ \|\vec{x} - \vec{a}\| < r\}$. (or "$\leq$"). Not trivial to prove!
(A ball is also an "interval"-esque set.)

• The set of solutions of $\vec{A}\vec{x} = \vec{b}$.
Proof: If $\vec{u}, \vec{v}$ are sol'ns, then $\vec{A}\underbrace{(\lambda\vec{u} + (1-\lambda)\vec{v})}_{\text{is therefore a solution.}} = \lambda\vec{b} + (1-\lambda)\vec{b}$

• Exercises: The set of $\vec{x}$; $\vec{A}\vec{x}$ is coordinate-wise $\leq \vec{b}$

• Note: The solutions of $\vec{A}\vec{x} = \vec{B}$: just like lin. comb's, lin. indep./dep., the concept works way beyond n-vectors!

Intersecting convex sets:

Claim: The intersection $S = S_1 \cap S_2$ of two convex sets S_1 and S_2, is a convex set.

Proof: Suppose $\vec{u} \in S$, $\vec{v} \in S$ and $\lambda \in (0,1)$.

Then $\lambda \vec{u} + (1-\lambda)\vec{v}$ is: $\begin{cases} \in S_1, \text{ as both points are } \in S_1 \text{ and } S_1 \text{ is convex} \\ \in S_2, \text{ same argument.} \end{cases}$

and thus $\in S_1 \cap S_2$.

The book calls this proof "One of the world's simplest". If you don't think it is simple, it is probably a language hurdle: draw a Venn diagram!

could be more than a sequence!

Exercise: Let $\{S_j\}_{j \in J}$ be an arbitrary collection of convex sets. Show that $\bigcap_{j \in J} S_j$ is convex.

Terminology: the convex hull of a set T is the intersection of all convex supersets.

Fact: it is the smallest convex $\underbrace{\text{superset}}_{\text{Possibly } = T}$ of T.

by prev. exercise

Ex.: T = this curve in $\mathbb{R}^2$: 

Convex hull:  drop-shaped.

Note: the union of convexes need not be convex. E.g.:  

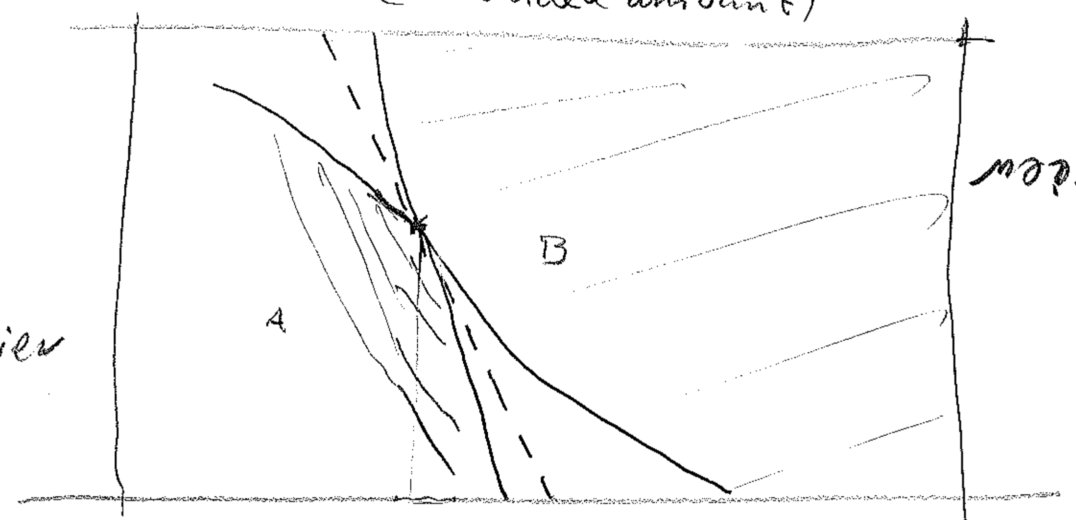
Why convex sets?

Separating hyperplanes & the 2^{nd} theorem of welfare economics

Convex preferences $\longleftrightarrow$ the set of $\vec{x}$ s.t $\underset{\text{week pref}}{\vec{x} \succsim \vec{x}^*}$ convex, each $\vec{x}^*$.

Edgeworth box, 2 goods, 2 agents
(in bounded amount)

Agent #1's view



Agent #2's view

Fact: Given two <u>disjoint</u> convex sets A, B, $\subseteq \mathbb{R}^n$,
A open. Then there is a $\vec{p}$
such that

$$\vec{p}^T \vec{u} < \vec{p}^T \vec{v}, \text{ all } \vec{u} \in A, \vec{v} \in B.$$

but works <u>way</u> more generally!

(Dashed line: $\vec{p}^T \vec{x} = m$, budget ...)

(The pricing functional $\vec{p}$ is not unique.
If boundaries are "smooth at tangency",
$\vec{p}$ is unique up to scaling.)

Convex preferences $\Longleftrightarrow$
quasiconcave utility functions

First: the more restrictive notion
of convex / concave functions.

Convex / concave functions

Note: "functions" here output numbers, like Math 2.

We shall give two defn's for convex functions.
First, a geometric.

Terminology: Epigraph: set of points on or above the graph.

$\text{epi}(f)$: the set $\{(\vec{x}, \vec{z});\ z \geq f(\vec{x})\}$.

Ex: $f(x) = \max\{0, x\}$  "fill up with water"

Note: one more dimension than $\vec{x}$.

Definition (I): convex function.

Let f be defined on some convex $S \subseteq \mathbb{R}^n$ (or: some vector space!)

f is convex if $\text{epi}(f)$ is a convex set.

and strictly convex if $\text{epi}(f)$ is a strictly convex set.

Note: The requirement that S be convex, follows automatically if we omit it.

Ex: $|x|$ is convex. Even $\|\vec{x}\|$ (not completely obvious!)

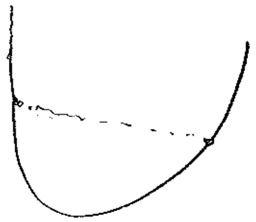
Definition (II), convex function (defined on convex S):

f is convex if, whenever $\vec{u} \in S$, $\vec{v} \in S$, $\lambda \in (0,1)$

$$f(\lambda \vec{u} + (1-\lambda)\vec{v}) \leq \lambda f(\vec{u}) + (1-\lambda) f(\vec{v}). \quad (*)$$

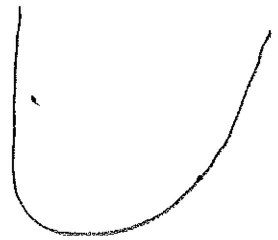
f is strictly convex if $(*)$ holds with $<$ except when $\vec{u} = \vec{v}$.

Def. II :



Pick two points on the graph. The connecting line segment is never below the graph.
("always above" for strict).

Def I



Pick two points on or above the graph. The connecting ...

Equivalent def's!

<u>Concave functions</u>:

<u>Def</u>: f is concave if $-f$ is convex.
f is strictly concave if $-f$ is strictly convex.

We can then formulate def'ns analogous to the above two.
Non-strict versions:

I: the convexity of the <u>hypograph</u>, i.e. the set of points on or <u>below</u> the graph

II: Whenever $\vec{u} \in S$, $\vec{v} \in S$ and $\lambda \in (0,1)$:

$$f(\lambda \vec{u} + (1-\lambda)\vec{v}) \geq \lambda f(\vec{u}) + (1-\lambda) f(\vec{v})$$

Note: For each pair $\vec{u}, \vec{v}$ of points – assume $\vec{u} \neq \vec{v}$ – consider $h(\lambda) = f(\lambda\vec{u} + (1-\lambda)\vec{v}) - \lambda f(\vec{u}) - (1-\lambda) f(\vec{v})$
"Convenient:" single variable $\lambda \in (0,1)$.

Similar for convex & for "strict"

II': f concave $\Leftrightarrow$ for each <u>pair</u> $\vec{u} \neq \vec{v}$ in S we have $h \geq 0$ on the interval $(0,1)$
(and... h concave in λ!)

We could consider convex comb's of several vectors:

f concave if

f (convex comb. of $\vec{v}^{(i)}$) $\geq$

convex comb. of the $f(\vec{v}^{(i)})$.

and even <u>Jensen's inequality</u>: f concave

"$\Leftrightarrow$" $f(E\vec{Y}) \geq E\, f(\vec{Y})$ "all" random vectors $\vec{Y}$

but then we would have to add a reservation like "as long as convergent or $= +\infty$ or $= -\infty$" ...

Jensen's inequality ties concavity to risk aversion: $f(EY) \geq Ef(Y)$ says "prefer (weakly) the expectation to the r.v.".

But, let's stick to def's I and II.

Proving concavity/convexity from def's could be demanding.

Ex: $f(x) = |x|$.

$h(\lambda) = |\lambda u + (1-\lambda)v| - \lambda|u| - (1-\lambda)|v|$.

No restriction to assume $u > v$

We have:

$h'(\lambda) = \underbrace{(u-v)\,\mathrm{sign}(\underbrace{\lambda u + (1-\lambda)v}_{\text{smallest at } \lambda = 0^+ \text{, since } v<u})}_{\text{if changes sign: from } - \text{ to } +.} - (|u| - |v|)$

So h' nondecreasing, with a possible upwards jump.

Since $h(0^+) = h(1^-) = 0$, we must "have the decreasing part of h first", so $h \leq 0$.

This idea works also for "Math 2 convex" functions.

Suppose $g'' \geq 0$. Let $v<u$.

Then $h(\lambda) = g(\lambda u + (1-\lambda)v) - \lambda g(u) - (1-\lambda)g(v)$

$h'(\lambda) = (u-v)g'(\quad) - g(u) + g(v)$

$h''(\lambda) = (u-v)^2 g''(\quad) \geq 0$.

So h' nondecreasing and h starts and ends at 0;

h must have the decreasing part first,

So $h \leq 0$ and g convex. (def II).

Properties ...? Characterization?

* Tempting to start <u>generally</u>, then impose conditions ... "if $f \in C^1$" or "if $f \in C^2$."
  (Shouldn't I then rather have started with quasiconvexes / quasiconcaves?)

* Alternative: C^2 first, then C^1, then ...

* Will do: First a few general properties that do not use derivatives (but could, if applicable)
  Then: characterization for C^2 functions
  Then: characterization for C^1 functions
  Then: characterization if not even C^1

If you prefer a different order — reshuffle the notes!

Three general facts: Concave?

1) If f and g are convex (on same domain) | concave

then $\max\{f(\vec{x}), g(\vec{x})\}$ is convex | $\underline{\min}\{f, g\}$ concave.

(Indeed: works for more than two functions!)

2) If f, g convex and $\overbrace{\alpha > 0, \beta > 0}^{\text{const}}$, | concave

then $\alpha f + \beta g$ convex | then concave

and <u>strictly</u> convex if f or g is | ditto!

[Q: different domains?]

3) Consider $F(\vec{x}) = h(f(\vec{x}))$

where f convex, takes values in $T \subseteq \mathbb{R}$

h convex and <u>nondecreasing</u> on T.

Then F is convex

SAME

Concave version: if

f, h <u>concave</u>

h nondecreasing

then F concave.

Before proving: how do 1) – 3) relate to Math 2?

1) the Math 2 case
well, in Math 2 we would need $\max\{f, g\}$ C^2 as well, so then it would be a nonneg 2^{nd} deriv.
(concave: $\min\{\dots\}$ ... nonpos ....)

2) the Math 2 case
At least for functions of a single variable,
$\alpha f'' + \beta g''$ ... ok!

3) the Math 2 case:
Easy for functions of a single variable

$$F' = h'(f(x))\, f'(x)$$

$$F''(x) = h''(f(x))\underbrace{(f'(x))^2}_{\geq 0} + \underbrace{h'(f(x))}_{\text{assumed } \geq 0} f''(x)$$

But generally?

Proof 1): intersect epigraphs!
Ex 1): $|x| = \max\{x, -x\}$ is convex!

Proof 2): Def'n II, apply ineq. for f and then g.
Strict case: one ineq strict!

Proof 3)(convex): $f(\lambda\vec{u} + (1-\lambda)\vec{v}) \leq \lambda \overbrace{f(\vec{u})}^{=s} + (1-\lambda)\overbrace{f(\vec{v})}^{=t}$

$h(f(\lambda\vec{u} + (1-\lambda)\vec{v})) \leq h(\lambda f(\vec{u}) + (1-\lambda)f(\vec{v}))$

since h nondecr. RHS $\leq \lambda h(s) + (1-\lambda)h(t)$ since h convex

Insert and we are done!

#3) is "dangerous":

You cannot "just turn everything upside down" to switch between concaves and convexes.

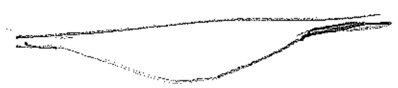
Ex: $e^{\|\vec{x}\|}$ is convex:

$\|\vec{x}\|$ is convex, and e^t is convex & increasing

Non-ex: $-e^{-\|\vec{x}\|}$ is <u>not</u> concave.

$-\|\vec{x}\|$ is concave, $-e^t$ is concave — but <u>not</u> nondecreasing!

$(-e^{-\|\vec{0}\|} = -1$ and as $\|\vec{x}\| \to +\infty$, $-e^{-\|\vec{x}\|} \to 0$, so it cannot possibly be concave.



C^2 functions, defined on convex $S \subseteq \mathbb{R}^n$.

Let $\vec{H} = \vec{H}(\vec{x})$ be the Hessian matrix

We have the implications

H pos. def on S except possibly at isolated points		... neg. ....
$\Downarrow$	note: only $\Downarrow$	$\Downarrow$
f strictly convex		f strictly concave
$\Downarrow$		$\Downarrow$
f convex		f concave
$\Updownarrow$	both ways	$\Updownarrow$
H pos. semidef on S		neg. semidef.

Note: · the "except possibly": x^4 is strictly convex, yet Hessian hits 0.
(impossible for quadratic functions, Hessian matrix is constant)

· the first "$\Downarrow$" is not a $\Updownarrow$.
det $\vec{H}$ could hit zero "more often than that" yet not destroying strict [concavity / convexity]

(corrected)

Convex:

Tangent **never above** the graph

- and touches only at that single point, for strictly convex functions.

1^{st} order approx: underestimates

Concave:

Tangent never below graph.

- ... strict...

1^{st} order approx around x^*: overestimates $f(x)$.

Concave/convex functions are "not that far from being piecewise differentiable".

For C' functions - drop the C^2 assumption

A function $f \in C'(S)$, S convex, open, $S \subseteq \mathbb{R}^n$,

is convex if and only if ⋮ concave

$\Updownarrow$

$f(\vec{x}) - f(\vec{x}^*)$

$\geq \nabla f(\vec{x}^*)\,(\vec{x} - x^*)$ $\leq$

all $\vec{x} \neq \vec{x}^*$

(and strictly so iff the ineq is strict, all $\vec{x} \neq x^*$).

Note: adding linear terms will not change concavity/convexity, so we can alternatively write:

- f convex iff the following holds:

For each $\vec{x}^* \in S$, the function g:

$g(\vec{x}) = f(\vec{x}) - f(\vec{x}^*) - \nabla f(\vec{x}^*)\,(x - x^*)$

"both or none convex"

has global min for $x = x^*$

- strictly if the min is always strict

For concave/strictly concave: "max"

Drop the C^1 assumption.

Fact: If the domain is open, a convex (/concave) function is continuous.

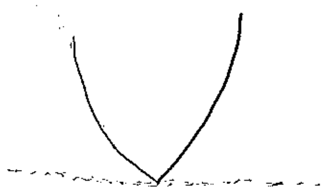
The only discontinuities can be on the boundary, (but they can be pretty bad:

Let $S = \{(x,y) ; x^2 + y^2 \leq 1\}$.

$f = \begin{cases} \text{nonnegative strictly concave on interior} \\ \text{anything} \leq 0 \text{, each boundary point} \end{cases}$

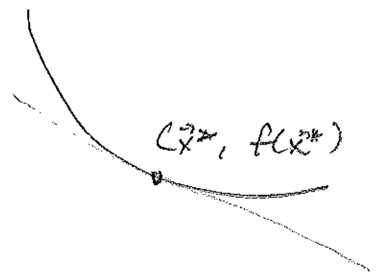
Yes, since S is a strictly convex set, we can draw $f(x,y)$ i.i.d. random ≤ 0 at boundary points!

So: Assume continuity. Concave / convex functions will be continuous on ECON4140



But: can we say something sensible about this situation?

Recall from the C^1 case:



Terminology: Given f on open convex set S

Fix $\vec{x}^* \in S$. Consider affine functions z:

$$\underbrace{z}_{z(\vec{x}),\ \text{affine}} - f(\vec{x}^*) = \vec{p} \cdot (\vec{x} - \vec{x}^*) \quad \text{in } \underbrace{S \times \mathbb{R}}$$

Same space as the graph of f. One dimension more!

Def:

- If $z \le f(\vec{x})$, all $\vec{x} \in S$, we call $\vec{p}$ a <u>subgradient</u> of f at $\vec{x}^*$
- The set of all such possible $\vec{p}$ s.t. $z \le f(\vec{x})$ is called the <u>subdifferential</u> of f at $\vec{x}^*$

Analogously: If $z \ge f(\vec{x})\ \forall \vec{x} \in S$: <u>supergradient</u>

and the possible $\vec{p}$'s: superdifferential

Note that this def'n is "pointwise", each $\vec{x}^*$

Facts:

- If $\vec{\nabla} f(x^*)$ exists, then it is the only possible sub-/supergradient at $\vec{x}^*$
- f convex $\Leftrightarrow$ has a subgradient at each $x^* \in S$
- concave $\Leftrightarrow$ has a super.....

(Here we still assume f defined on open convex S.

What did this mean ... ?

→ try to form subgradients for an arbitrary continuous f.

If f is not convex, you will fail somewhere ...

because z was required to be ≤ f everywhere, (each $\vec{x}^*$)

(This formulation of subgradients / subdiff. / super- ....

is "tailored to the narration" ... where the point is to grasp the geometric behaviour.)

So we have generalized the derivative.

Now generalizing stationary points ... ?

Note: a local min for a convex, is global

a local max .... concave

Fact: Let f be defined on an open convex S.

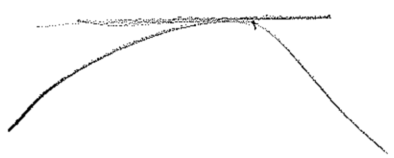
If $\vec{0}$ is a subgradient at x^*,

then x^* is global min

If $\vec{0}$ is super ....

global max

"Proof": Put a horizontal hyperplane atop the graph.



If that is possible – without cutting through the graph elsewhere – then ...

Also note:

The set of global $\underset{\text{max}}{\text{min}}$ of a $\underset{\text{concave}}{\text{convex}}$ f

is <u>convex</u>.

(A "flat plateau atop a concave" must form a convex subset of S.
– this is a special case of <u>quasi</u>concavity)

So... st. pts....?

C^1 functions: set $\nabla f = \vec{0}$

Convex/concave ... set ... what?

Sorry. Easily becomes inconvenient by hand (but not hopeless to implement numerically)

Simple example: $|x| + |y-3|$

$\nabla f = (\text{sign } x, \text{sign}(y-3))$ whenever well-defined

At $x=0$, $\frac{\partial f}{\partial x}$ crosses 0. At $y=3$, $\frac{\partial f}{\partial y}$ crosses 0.

So $\vec{0}$ <u>does</u> the subgradient job at $(0,3)$.

One more characterization:

Suppose f continuous on convex S.

If for every pair $\vec{u}, \vec{v}$ in S, $\vec{u} \neq \vec{v}$

we have

$$f\left(\tfrac{1}{2}(\vec{u}+\vec{v})\right) \leq \tfrac{1}{2}\left(f(\vec{u})+f(\vec{v})\right)$$

(resp $<$ resp $\geq$ resp $>$)

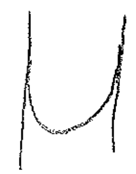
Embarassing :-/
Mixed up
concavity /
convexity.

then f is convex resp strictly convex
resp concave resp strictly concave

So if we know we have continuity, then
the unweighted average $\lambda = \frac{1}{2}$ suffices!

Q: Is there anything "lost" by assuming
"S open and convex" rather than
"S convex, f continuous" ?

A: Well... kinda...



vertical tangents.

Open S: tangents may $\longrightarrow$ vertical
slopes may $\longrightarrow +\infty$ or $-\infty$

boundary: could <u>be</u> vertical / $\pm\infty$.

Do not bother.

Convex/concave functions of a single variable are integrals of their "derivatives".

Consider the following property,

($\circledast$) Fix open interval $S = (a, b)$. There is a g such that for all α, β in S:

$$\int_{\alpha}^{\beta} g(x)\,dx = f(\beta) - f(\alpha).$$

Facts:

f convex on (a, b) $\Leftrightarrow$ $\circledast$ holds with some nondecreasing g.

strictly ..... strictly increasing

concave ...... nonincreasing

strictly concave .... strictly decreasing.

(g could have infinitely many discontinuities, — for example, it could jump at every rational number!)

... and "recall": f convex $\Leftrightarrow$ for any pair $\vec{u}, \vec{v}$ the function $h(\lambda) = f(\lambda\vec{u} + (1-\lambda)\vec{v}) - \lambda f(\vec{u}) - (1-\lambda) f(\vec{v})$ is convex on $\lambda \in [0, 1]$