1st order linear diff eq. systems in $\mathbb{R}^2$ $\longrightarrow$ 2nd order in $\mathbb{R}$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \vec{A} \begin{pmatrix} x \\ y \end{pmatrix} + \vec{b}(t)$$

$\vec{A}$ = matrix of constants.

NOTE : $\ddot{x} + a\dot{x} + bx = 0$

$\dot{\vec{x}} = \vec{A}\vec{x} + \vec{b}$

sometimes confusing notation!

Shall :
- → rewrite the system to a 2nd-order in $x$
- → characteristic equation for that eq
- ⇔ — " — for $\vec{A}$ . Eigenvalues!

We have

$$\dot{x} = a_{11} x + a_{12} y + b_1(t)$$
$$\dot{y} = a_{21} x + a_{22} y + b_2(t)$$

(If $a_{12} = 0$, solve first for $x$, then insert, solve for $y$.
If $a_{21} = 0$ . . . . . . . . . $y$ . . . . . . . . . . $x$.)

(The following approach will also work - but with more/uglier terms - if $\vec{A} = \vec{A}(t)$. Skip that!)

Idea: $\frac{d}{dt}\dot{x} = \frac{d}{dt}[a_{11}x + a_{12}y + b_1]$ yields a third lin. eq. Eliminate $y$ and $\dot{y}$.

If $a_{12} \neq 0$:

$$\ddot{x} = a_{11}\dot{x} + a_{12}\dot{y} + \dot{b}_1(t)$$

$$= a_{11}\dot{x} + a_{12}\left[a_{21}x + a_{22}y + b_2\right] + \dot{b}_1$$

Now $a_{12}y = (\dot{x} - a_{11}x - b_1)\cancel{/a_{12}}$, so

$$\ddot{x} = a_{11}\dot{x} + a_{12}a_{21}x + a_{22}\dot{x} - a_{11}a_{22}x - a_{22}b_1 + \dot{b}_1$$
$$- a_{12}b_2$$

$$= (\mathrm{tr}\,A)\dot{x} - (\det A)x + a_{12}b_2 - a_{22}b_1 + \dot{b}_1$$

→ If we did compute $\ddot{y}$ instead we would get same homogeneous!

Now $y = \dfrac{\dot{x} - a_{11}x - b_1}{a_{12}}$

$$r^2 - (\mathrm{tr}\,A)r + \det A$$

$$\begin{vmatrix} a_{11}-r & a_{12} \\ a_{21} & a_{22}-r \end{vmatrix} = (a_{11}-r)(a_{22}-r) - a_{12}a_{21}$$
$$= r^2 - r(\mathrm{tr}\,A) + \det A.$$

So ~~with~~ with $x = C_1 u_1(t) + C_2 u_2(t) + u^*$

then $y = P_1 u_1(t) + P_2 u_2(t) + v^*$

with $$v^* = \frac{\dot{u}_1^* - a_{11} u^* - b_1}{a_{12}}$$

and $P_1, P_2$ depend on $C_1, C_2$ as follows
[look up the book]

→ two distinct real roots / ~~eigenvalues~~ $\lambda_1, \lambda_2$: $C_i e^{\lambda_i t}$

$$P_i = C_i \frac{\lambda_i - a_{11}}{a_{12}} \qquad i = 1, 2$$

→ double root: $(A + Bt) e^{\lambda t}$

$$P_1 = \frac{\lambda - a_{11}}{a_{12}} A + \frac{B}{a_{12}} \quad (\cancel{\text{as above}})$$

$$P_2 = \frac{\lambda - a_{11}}{a_{12}} B \quad \text{as above}$$

→ two non-real: $\lambda_1 = \alpha + i\beta \qquad \lambda_2 = \alpha - i\beta$

$$P_1 = \frac{\alpha - a_{11}}{a_{12}} C_1 + \frac{\beta}{a_{12}} C_2$$

$$P_2 = \ldots C_2 - \ldots C_1$$

ALTERNATIVE APPROACH

If $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \bar{A}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} b_1(t) \\ b_2(t) \end{pmatrix}$

($\bar{A}$ constant wrt. $t$)

then
$$\ddot{x} - (\mathrm{tr}\,\bar{A})\,\dot{x} + (\det \bar{A})\,x = f_1(t) \qquad (*)$$
$$\ddot{y} - (\mathrm{tr}\,\bar{A})\,\dot{y} + (\det \bar{A})\,y = f_2(t)$$

with $f_1(t) = a_{12}\, b_2(t) - a_{22}\, b_1(t) + \dot{b}_1(t)$.

$\to$ If $a_{12} = 0$ then $\dot{x} = a_{11} x + b_1$

solve this, plug into

$$\underbrace{\dot{y} - a_{22} y}_{1^{st}\text{ order}} = a_{21} x + b_2$$

and solve.

$\to$ If $a_{21} = 0$: solve first for $y$, then for $x$

Suppose $a_{12} a_{21} \neq 0$.

① Solve $(*)$ for $x = C_1 u_1 + C_2 u_2 + u^*$

② Then
$$y = \frac{\dot{x} - a_{11} x - b_1}{a_{12}}$$

is likely easiest! (Even if you must calculate $\dot{x}$)

Ex:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-\pi t} \qquad (\pi \approx 3.14159).$$

Eigenvalues: $\underbrace{1}_{\alpha} \pm i \underbrace{\sqrt{3}}_{\beta}$, complex.

$$x(t) = C_1 e^t \cos(t\sqrt{3}) + C_2 e^t \sin(t\sqrt{3}) + u^*$$

where $u^*$ solves

$$\ddot{u}^* - 2\dot{u}^* + 4u^* = 3 \cdot 2e^{-\pi t} - e^{-\pi t} - \pi e^{-\pi t}$$

Try $K e^{-\pi t}$ to

$$(5-\pi) e^{-\pi t} = K e^{-\pi t} (\pi^2 - 2(-\pi) + 4)$$

$$K = \frac{5-\pi}{\pi^2 + 2\pi + 4}$$

Since $\dot{x} = e^t \cdot (C_1 \cos + C_2 \sin)$

$$+ e^t (C_1 \sqrt{3} (-\sin) + C_2 \sqrt{3} (\cos))$$

$$- K\pi e^{-\pi t}$$

$$\dot{x} - \underbrace{a_{11}}_{=1} x = [\text{these terms}] - K e^{-\pi t}$$

$$y = \frac{\sqrt{3}}{3} e^t \left[ C_2 \cos(t\sqrt{3}) - C_1 \sin(t\sqrt{3}) \right] - \frac{K(1+\pi)+1}{3} e^{-\pi t}$$

!

Special cases:

a) If $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ does not depend on $t$,

and $|\bar{A}| \neq 0$

then $\begin{pmatrix} u^* \\ v^* \end{pmatrix} = -\bar{A}^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

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* Let $\underline{\bar{x} \in \mathbb{R}^n}$, solve $\dot{\bar{x}} = \bar{A}\,\bar{x} + \bar{b}$

$\bar{A}$ have $n$ distinct (complex!) eigenvalues $\lambda_1 \dots \lambda_n$ with eigenvectors $\bar{v}_1 \dots \bar{v}_n$,

- homogeneous eq has general solution

$$C_1 \bar{v}_1 e^{\lambda_1 t} + \dots + C_n \bar{v}_n e^{\lambda_n t}$$

- if $\bar{b}$ does not depend on $t$ and $|\bar{A}| \neq 0$: particular solution $\bar{A}^{-1}\bar{b}$

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