Linear systems $\quad \dot{\vec{x}} = \vec{A}\vec{x} + \vec{f}(t)$, $n=2$

-what do we have?

* Can solve! Provided $\vec{f}$ nice enough.

  Roots = eigenvalues of $\vec{A}$

  If these are non-real: can still solve!

  (get: if $\lambda = tr\vec{A} \pm \sqrt{-\beta^2}$,

  then $Re\,\lambda = tr\vec{A}$ "into the exponent"

  and β "into the trig.".)

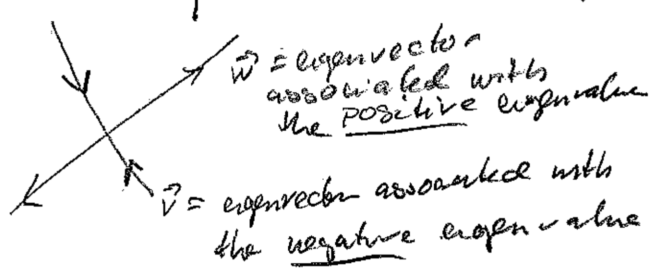
* Stability and (if $\vec{f}=0$) graphing the phase diagram

  → if $\min\{\lambda_1, \lambda_2\} > 0$: "source", unstable

  → if $\max\{\lambda_1, \lambda_2\} < 0$: "sink", asympt. stable

  → if opposite signs: <u>saddle point</u>: unstable,

  but two trajectories converge



The convergent path has slope $= \frac{x_2(t)}{x_1(t)} = \frac{v_2}{v_1}$

→ if $\lambda_1 > 0 = \lambda_2$: $Ce^{\lambda_1 t} + D$ unstable

→ if $\lambda_1 < 0 = \lambda_2$: —— " —— stable (not asympt.)

→ if $\lambda_1 = \lambda_2 = 0$: $At + B$ unstable

→ if $\lambda \notin \mathbb{R}$: spiralling

→ outwards if $\operatorname{tr} \vec{A} > 0$ unstable

→ inwards if $\operatorname{tr} \vec{A} < 0$ asympt. stable

→ ellipses if $\operatorname{tr} \vec{A} = 0$ stable (not asympt).

Next: <u>Nonlinear systems</u>

→ approximate with linear locally near equilibrium points

→ stability properties ≈ the linearized

although: • if $\max\{\lambda_1, \lambda_2\} = 0$ we cannot conclude on stability or not

• if $\min\{\lambda_1, \lambda_2\} = 0$: unstable, but we cannot conclude saddle or not

(• if $\lambda_1 = \lambda_2$: cannot tell whether it spirals)

Also: sketching phase diagrams.

ODE systems & stability

System: $\dot{\vec{x}} = \vec{F}(\vec{x})$ (autonomous, no explicit "t")

Def. Equilibrium point, a.k.a. stationary state $\vec{x}^*$,

One for which $\vec{F}(\vec{x}^*) = \vec{0}$.

i.e.: $\vec{x}(t) \equiv x^*$ (constant!) is a particular solution.

—

Stability.

Idea: "stable", can keep $\vec{x}(t)$ close to $\vec{x}^*$ by merely starting at $\vec{x}(t_0)$ sufficiently close to $\vec{x}^*$.

"asymptotically" stable: also converges

Def'n: next page.

For nonlinear systems, these are "local" properties.

For linear systems: we "lose a few nuances" by thinking of them as "global".

| Ex: $\dot{x} = mx$, $\dot{y} = 0$ stationary on the entire y axis

Most important in Math 3:
loc. asympt. stable ? unstable ?
And if unstable: saddle point ?

Def. A stationary state $\vec{x}^*$ is

* <u>stable</u> if for every $\varepsilon > 0$ (max allowed excursion)

there is some $\delta > 0$ such that

whenever $\vec{x}$ starts within distance δ from $\vec{x}^*$

i.e.: $\|\vec{x}(t_0) - \vec{x}^*\| < \delta$

we will have $\|\vec{x}(t) - \vec{x}^*\| < \varepsilon$, all $t \geq t_0$

* unstable if not stable

* locally asymptotically stable if stable and furthermore, some $\delta > 0$, if

$$\|\vec{x}(t_0) - \vec{x}^*\| < \delta$$

then $\|\vec{x}(t) - \vec{x}^*\| \to 0$ as $t \to +\infty$.

[Minor detail: could be that this "δ" is less than the δ chosen in the stability def'n — but we can then always choose the smallest of the two]

* <u>globally</u> asymptotically stable if stable and

$$\|\vec{x}(t) - \vec{x}^*\| \to 0 \text{ as } t \to +\infty$$

for <u>all</u> solutions $\vec{x}(t)$.

* For $n = 2$: <u>saddle point</u> if unstable and some $\vec{x}(t) \to \vec{x}^*$ without starting there.

Eigenvalue characterization

* Linear case: $\dot{\vec{x}} = \vec{A}\vec{x}$, ($\vec{A}$ constant)

(−) If all eigenvalues have <u>negative</u> <u>real</u> <u>parts</u> [note: if $\lambda \in \mathbb{R}$, then $Re(\lambda) = \lambda$]:

asymptotically stable

(+) If <u>some</u> eigenvalue has <u>positive</u> real part:
unstable.

(S) $n=2$: If $\lambda_1 > 0 > \lambda_2$: saddle point.
(unstable)

For the linear case, we <u>can</u> classify also if the largest real part is 0, but ... skip it.

** Nonlinear case.

For <u>each</u> stationary state $\vec{x}^*$:

- Calculate the Jacobian of $\vec{F}$ at $\vec{x}^*$:
$$\vec{J} = \frac{\partial \vec{F}}{\partial \vec{x}}(x^*)$$
- Classify the <u>linear</u> system $\dot{\vec{z}} = \vec{J}\vec{z}$
according to (−) / (+) / (S) above.
$\Downarrow$ loc. asympt. stable $\Downarrow$ unstable $\Rightarrow$ saddle
- If $\max_i Re(\lambda_i) = 0$: <u>no conclusion</u>!
- $n=2$: If $\lambda_1 > 0 = \lambda_2$: unstable, but we don't know if saddle.

For tomorrow:

$$\begin{pmatrix}\dot x\\ \dot y\end{pmatrix} = \bar M \begin{pmatrix}\frac{1}{2}(x^2-1)\\ y\end{pmatrix}$$

assume $|\bar M| \neq 0$

Nonlinear!

Jacobian: $\bar J = \bar M \begin{pmatrix} x & 1\\ x & 1\end{pmatrix}$

(check for yourselves!)

Eq. pts: $\bar M \begin{pmatrix}\frac{1}{2}(x^2-1)\\ y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$\Longleftrightarrow$ since $|\bar M| \neq 0$ $\quad x^2=1,\ y=0 \quad$ $(-1,0)$ and $(1,0)$

$$\bar J(\pm 1, 0) = \bar M \begin{pmatrix}\pm 1 & 1\\ \pm 1 & 1\end{pmatrix}$$

$$= \begin{pmatrix}\pm(m_{11}+m_{12}) & m_{11}+m_{12}\\ \pm(m_{21}+m_{22}) & m_{21}+m_{22}\end{pmatrix}$$

has determinant $=0$ (proportional columns!)

$$\begin{vmatrix}\pm p - \lambda & p\\ \pm q & q-\lambda\end{vmatrix} = \lambda^2 - \lambda(q \pm p)$$

("+" for $(1,0)$, "−" for $(-1,0)$)

eigenvalues $\lambda_1 = 0$ and $\lambda_2 = q \pm p$

$(1,0)$: unstable if $q+p>0$, no conclusion otherwise

$(-1,0)$ unstable if $q-p>0$, —— " ——

Graphical tool: phase plane analyses.

$f(x,y) = 0$: curve in (x,y)-plane

→ at the curve: $\dot{x} = 0$ i.e. movement ↓ or ↑ or •

$g(x,y) = 0$: analogous. No movement in y-direction

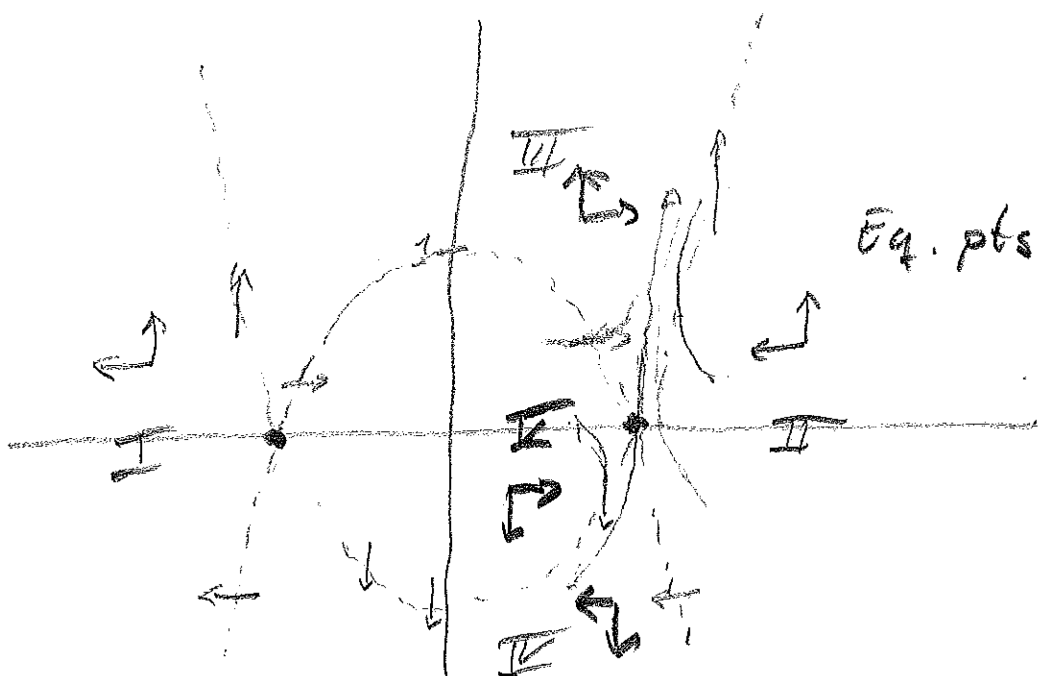
These curves are called nullclines.

Ex:

$$\dot{x} = y - x^2 + 1$$
$$\dot{y} = y + x^2 - 1$$

zero when $y = x^2 - 1$

$y = 1 - x^2$



Eq. pts: $(\pm 1, 0)$

Nullclines partition the plane. In this case into five domains

Above $\frown$: $\dot{y} > 0$. "Upwards"

Below — " — : downwards

Above $\smile$: $\dot{x} > 0$: right

Below $\frown$: $\dot{x} < 0$: left