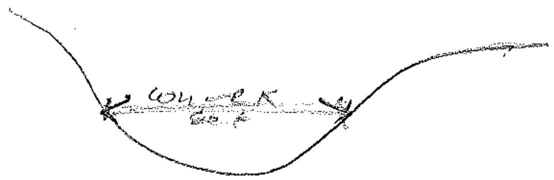
Quasiconcave / quasiconvex functions

Convex vs Quasiconvex



* Fill up with water;
  the water is a convex set.
* Equivalent, but a bit more involved: For any level ℓ:
  - Fill up to level ℓ 
  - The water is a convex set

For any level ℓ:
- Fill up to level ℓ
- The water surface (bird's view!) is a convex set.

So:
This set is convex:
$\{(z, \vec{x}) \in \mathbb{R}^{n+1};\ f(\vec{x}) \leq z\}$

Def I: f quasiconvex if:
These sets are convex:
$\{\vec{x} \in \mathbb{R}^n;\ f(\vec{x}) \leq z\}$
for every z!

The respective definition requires the connecting line to be above the graph whenever we pick two points that are:

or at

convex	quasiconvex
above (or on) the graph	above (or on) the graph AND at same (vertical) level.

In particular, every convex function is also quasiconvex.

This definition is convenient to show that the max of two quasiconvex functions is quasiconvex [But: the sum need not be!]

But let us turn to quasiconcave, as you probably see more of those:

Def: f quasiconcave if $-f$ is quasiconvex

(Follows: min{two quasiconcaves} is quasiconcave)

Probably this is easier to relate to microeconomics:

Def II: A function f defined on a convex set S is quasiconcave if for any two $\vec{u} \neq \vec{v}$ in S, any $\lambda \in (0,1)$ we have

$$f(\lambda \vec{u} + (1-\lambda)\vec{v}) \geq \min\{f(\vec{u}), f(\vec{v})\} \quad (*)$$

Utility interpret.: a weighted avg. is better than the worst. (or equal)

Or: moving towards a better vector, improves from step one. (No "one step back in order to get two steps forward")

Strict: $(*)$ holds with ">".

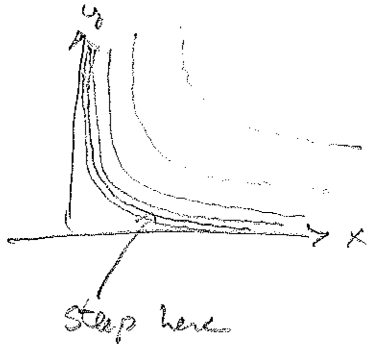
(Note $\vec{u} \neq \vec{v}$ assumed, and $\lambda \notin \{0,1\}$.)

Quasiconcavity is preserved under increasing transformations.

Strict quasiconcavity: under strictly incr. transf'ns

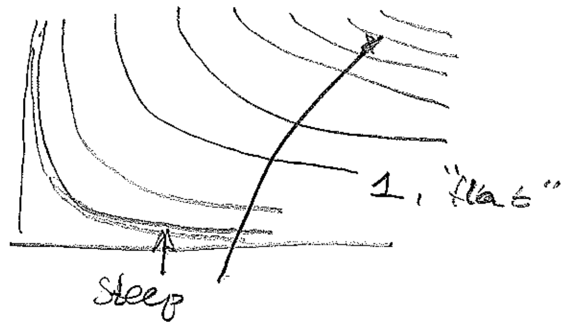
Ex:

Cobb-Douglas $g(x,y)$



Concave

$f(x,y) = e^{(g(x,y)-1)^3}$



Quasiconcave.

Same level curves, different levels.

Remember: concave

$f = h(g(\vec{x}))$

incr

is concave

quasiconcave

$f(h(g(\vec{x})))$

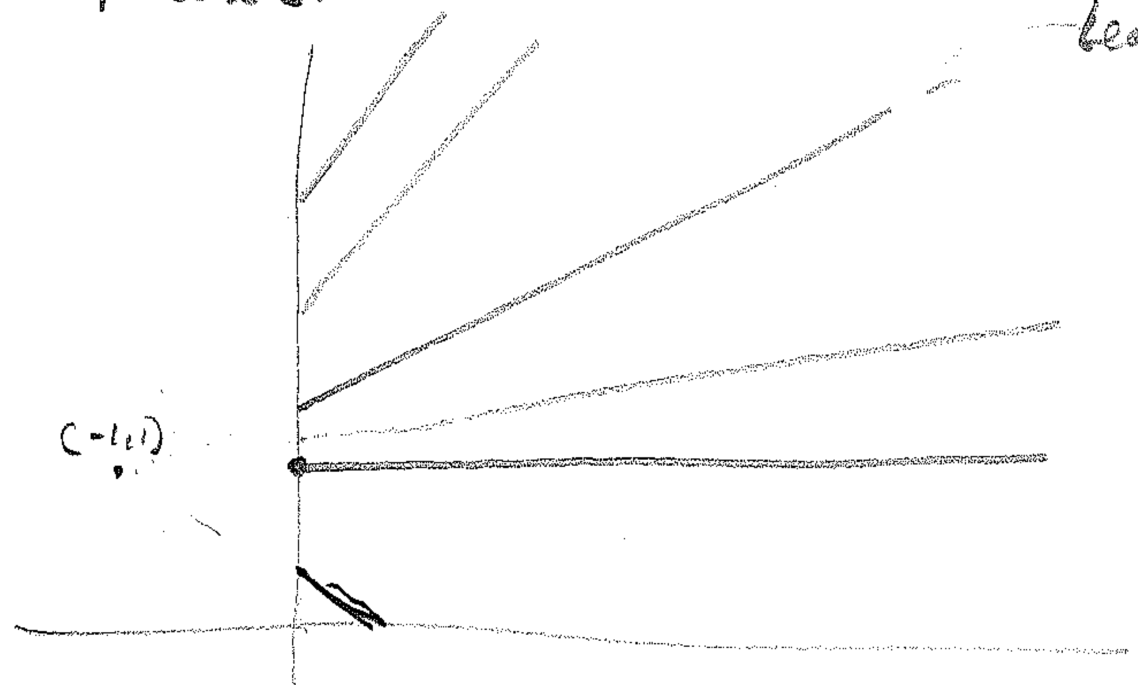
increasing ($\Rightarrow$ quasi-concave!)

is quasiconcave.

Def: <u>Quasilinear</u>: both quasiconcave and quasiconvex.

Ex: Any monotonous function of a single variable. (Could have jumps!)

Example: Define f for all $x \geq 0$, $y \geq 0$ as follows:



For $(x,y) \in \mathbb{R}^2_+$, consider the line from (x,y) to $(-1,1)$. Let $f(x,y) = 1 +$ slope of this line.

Then f = the y-coord of where the level curve hits the 2^{nd} axis.

$$f(x,y) = \frac{x+y}{x+1}$$

Quasilinear!

Upper level sets: $f(x,y) \geq L \Leftrightarrow$

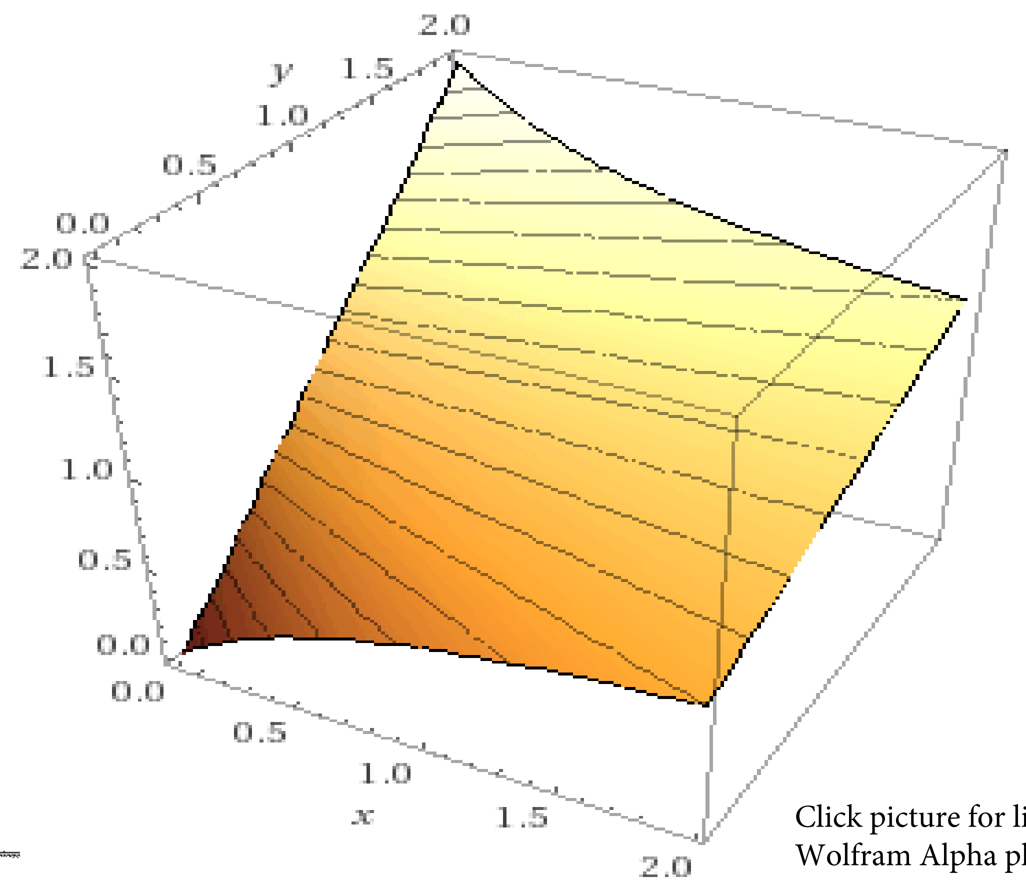
$(x,y) \in$ half-plane above line $\cap \mathbb{R}^2_+$

convex $\cap$ convex.

Lower: half-plane below line $\cap \mathbb{R}^2_+$

Warning: This f is <u>not</u> a transformation of a concave or a convex f.

A concave/convex <u>cannot</u> produce such level curves!



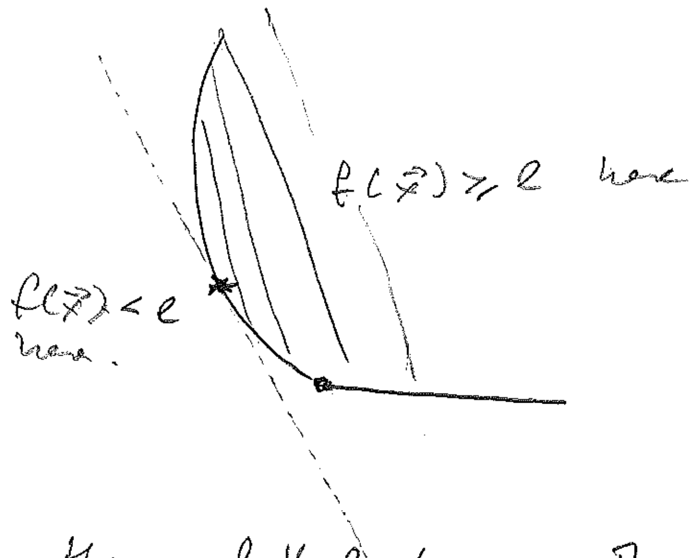
Click picture for link to Wolfram Alpha plot

Quasiconcave functions need not be "nice" at all. Ex: Let $g(\vec{x})$ be Cobb-Douglas, and let

$$f(\vec{x}) = \begin{cases} g(\vec{x}) & \text{if } g(\vec{x}) < 1 \\ [\text{draw random i.i.d.} \in [1,2] \text{ for each } \vec{x} \text{ s.t. } g(\vec{x}) = 1] & \\ g(\vec{x}) + 2 & \text{if } g(\vec{x}) \in (1,2) \\ [\text{draw} \ldots \in [4,5] \text{ if } g(\vec{x}) = 2] & \\ g(\vec{x}) + 5 & \text{if } g(\vec{x}) > 2 \end{cases}$$

Nevertheless, some characterizations for C^2/C^1 functions are interesting.

Preliminary: ($\vec{x} \in \mathbb{R}^2$ on the sketch)



Constrain f to the dotted line. The maximum subject to that line, is the $*$ point!

In fact: if for any such tangent [mind the cusp at • !]

we have max @ tangency point, then f is quasiconcave!

C^2 characterization for strict quasiconcavity.

The tangent hyperplane is now orthogonal to ∇f. Let $\vec{H} = \vec{H}(\vec{x})$ be the Hessian.

corr. 2019

Fact: if for every $\vec{x}^*$ we have $\vec{H}(\vec{x}^*)$ ~~pos.~~ **negative** def subject to the constraint $\vec{p}^T \vec{z} = 0$ where $\vec{p}^T = \nabla f(\vec{x}^*)$,

then f is strictly quasiconcave.

Do we have a "$\Longleftrightarrow$"? No. $\vec{H}$ and ∇f could hit zero.

Sufficient: $(-1)^r b_r > 0$, $r = 2, \ldots, n$

where $b_n = \begin{pmatrix} 0 & \nabla f(x^*) \\ \nabla f(x^*)^T & H(x^*) \end{pmatrix}$

and b_r is the $(r+1) \times (r+1)$ leading principal minor.

For $n = 2$: $\begin{vmatrix} 0 & f'_x & f'_y \\ f'_x & f''_{xx} & f''_{xy} \\ f'_y & f''_{yx} & f''_{yy} \end{vmatrix} > 0$

$\Rightarrow$ strict quasiconcavity.

That is 3x3, but n=2;
r runs from 2 to 2,
and $(-1)^r = (-1)^2 = 1$

Example: $f(x,y) = xy$ on:

(a) open first quadrant: On $\{(x,y) ;\ x>0, y>0$

(b) open second quadrant: On $\{(x,y) ;\ x>0>y\}$.

$\nabla f(x,y) = (y, x)$ $\qquad \begin{vmatrix} 0 & y & x \\ y & 0 & 1 \\ x & 1 & 0 \end{vmatrix} = 2xy.$

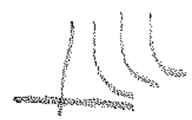
(a) Strictly quasiconcave when $2xy > 0$.

(b) Not; but same for $-f$, yields $\underbrace{-2x}_{>0}\underbrace{y}_{>0} > 0$

So f is strictly quasiconvex on the second quadrant

Note:

The example was an illustration of the criterion. Arguably faster:

$f \geq L \Leftrightarrow \quad y \geq L/x$  convex sets

$\leq L$: 

Example: Let a, b be >0.

$f(x,y) = x^a y^b$ on $\mathbb{R}_+^2$. Quasiconcave

since upper level sets

$y \geqslant \text{constant} \cdot x^{-a/b}$ are convex.

convex function

But to illustrate the determinant criterion:

$$f'_x = \frac{a}{x} f \quad ; \quad f'_y = \frac{b}{y} f$$

$$f''_{xx} = \frac{a(a-1)}{x^2} f \; ; \; f''_{xy} = \frac{ab}{xy} f \; ; \quad f''_{yy} = \frac{b(b-1)}{y^2} f$$

$$\left| \begin{pmatrix} 0 & \frac{a}{x} & \frac{b}{y} \\ a/x & \frac{a(a-1)}{x^2} & \frac{ab}{xy} \\ b/y & \frac{ab}{xy} & \frac{b(b-1)}{y^2} \end{pmatrix} f \right| = \frac{abf^3}{xy} \begin{vmatrix} 0 & 1 & 1 \\ a/x & \frac{a-1}{x} & \frac{a}{x} \\ b/y & \frac{b}{y} & \frac{b-1}{y} \end{vmatrix}$$

$$= \left(\frac{ab}{xy}\right)^2 f^3 \begin{vmatrix} 0 & 1 & 1 \\ 1 & (1-\frac{1}{a}) & 1 \\ 1 & 1 & (1-\frac{1}{b}) \end{vmatrix} = \left(\frac{ab}{xy}\right)^2 f^3 \cdot \left(1 - (1-\tfrac{1}{b}) + 1 - (1-\tfrac{1}{a})\right)$$

$$= \frac{ab}{(xy)^2} f^3 \cdot (a+b) \quad .$$

The matrix

$$\begin{bmatrix} 0 & \nabla F \\ (\nabla F)^T & \text{Hesse}[F] \end{bmatrix}$$

(and its determinant)

are called the

<u>BORDERED HESSIAN of F.</u>

One application: Let $n = 2$. Then

$$\begin{vmatrix} 0 & F'_x & F'_y \\ F'_x & & \\ F'_y & & \text{Hesse} \end{vmatrix} = -\left((F'_x)^2 F''_{yy} - 2F'_x F'_y F''_{xy} + (F'_y)^2 F''_{xx}\right)$$

Now the <u>elasticity of substitution</u>

3. The elasticity of substitution defined in (2) can be expressed in terms of the derivatives of the function F:

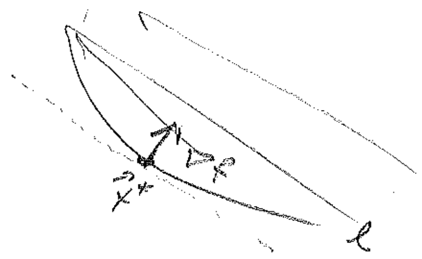
$$\sigma_{yx} = \frac{-F_1' F_2'(xF_1' + yF_2')}{xy[(F_2')^2 F_{11}'' - 2F_1' F_2' F_{12}'' + (F_1')^2 F_{22}'']}, \qquad F(x, y) = c$$

Use this formula to derive the result in Example 2.

equals

$$\frac{\frac{x}{F'_x} \; \frac{y}{F'_y} \; \nabla F \cdot \begin{pmatrix} x \\ y \end{pmatrix}}{|\text{bordered Hessian}|}$$

When utility / production is increasing & quasiconcave, "all terms here" are ≥ 0, C' characterization



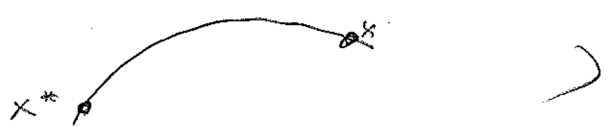
* f quasiconcave
$\Leftrightarrow$ for any two $\vec{x}$, $\vec{x}^*$ with $f(x) \geq f(x^*)$
we have $\nabla f(\vec{x}^*)(\vec{x} - \vec{x}^*) \geq 0$

* If furthermore $\nabla f(\vec{x}^*)(\vec{x} - \vec{x}^*) > 0$
except when $\vec{x} = \vec{x}^*$, then f
is strictly quasiconcave.
[Note: no "$\Leftrightarrow$"; counterex: $f(x) = x^3$ (at 0).]

Interpretation: Recall that
$$\nabla f(\vec{x}^*) \frac{\vec{x} - \vec{x}^*}{\|\vec{x} - \vec{x}^*\|}$$ is the
<u>directional derivative</u> in the direction
towards the "better" point $\vec{x}$.
(if more is better)

First step towards something better, improves!

(This does not say that f increases monotonously
when moving from $\vec{x}^*$ to $\vec{x}$:



Quasiconcave / quasiconvex homogeneous positive functions

Fact: Let f be defined on a convex cone K.
[recall: a cone satisfies that $\vec{x} \in K \Downarrow t\vec{x} \in K \ \forall t \geq 0$]

Suppose that $f(\vec{x}) > 0$ if $\vec{0} \neq \vec{x} \in K$,
[it will follow that $f(\vec{0}) = 0$ if f defined there]

and that f is (positive-) homogeneous of degree $q > 0$: $f(t\vec{x}) = t^q f(\vec{x})$, all $t \geq 0$ all $\vec{x}$.

Then:

* If f is quasiconcave and $q \in (0,1]$ then f is concave.

* If f is quasiconvex and $q \geq 1$ then f is convex.

Exercise: Suppose we have proven the case $q = 1$. Why does the rest ($q \in (0,1)$ resp $q > 1$) follow? Show that!

The proof $(q=1)$: Fix $\vec{u} \in k$, $\vec{v} \in k$. Consider

$f(\lambda \vec{u} + (1-\lambda)\vec{v})$.

Exercise: Show that everything is ok if $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$!

The case $\underbrace{\vec{u} \neq \vec{0} \neq \vec{v}}_{\Rightarrow f(\vec{u}) \cdot f(\vec{v}) > 0}$, rough sketch:

Write $\vec{u}$ as $\frac{f(\vec{u})}{f(\vec{v})} \cdot \underbrace{\frac{f(\vec{v})\vec{u}}{f(\vec{u})}}_{=:\vec{w}}$, Then $f(\vec{w}) = f(\vec{v})$, by homogeneity.

and $\lambda \frac{f(\vec{u})}{f(\vec{v})} \vec{w} + (1-\lambda)\vec{v}$ is a weighted sum

of $\vec{w}$ and $\vec{v}$, that is:

$S \cdot \underbrace{(t\vec{w} + (1-t)\vec{v})}_{\text{weighted average, convex comb.}}$

$S = \lambda \frac{f(\vec{u})}{f(\vec{v})} + (1-\lambda)$ is the sum of weights.

$f(\text{this}) = S\, f(t\vec{w} + (1-t)\vec{v})$ (homogeneity)

$\begin{cases} \leq S \max\{f(\vec{w}), f(\vec{v})\} & \text{if } f \text{ quasiconvex} \\ \geq S \min\{f(\vec{w}), f(\vec{v})\} & \text{if } f \text{ quasiconcave} \end{cases}$

Since $f(\vec{w}) = f(\vec{v})$, both the max and the min equal $t f(\vec{w}) + (1-t) f(\vec{v})$

Now insert, and get $\lambda f(\vec{u}) + (1-\lambda) f(\vec{v})$.

Example $f(\vec{x}) = x_1^{a_1} \cdot \ldots \cdot x_n^{a_n}$, defined where all $x_i > 0$,

where each $a_i > 0$, and $\sum_i a_i \leq 1$:

concave.

This example highlights several crucial properties:

→ $g(\vec{x}) := \ln f(\vec{x}) = \underbrace{\sum_i a_i \underbrace{\ln x_i}_{\text{concave}}}_{\text{positive scaling of concaves}}$ is the sum of concave functions

→ $f(\vec{x}) = e^{g(\vec{x})}$ exp increasing.

· Recall: what transformations of a concave/convex yield concave/convex/quasiconcave/quasiconvex?

→ f is quasiconcave and homogeneous of degree $\sum_i a_i \leq 1$, and $f > 0$ on the set $\{\vec{x};$ all $x_i > 0\}$. $\Rightarrow$ Concave there.

(That f is even concave on $\{\vec{x};$ all $x_i \geq 0\}$: continuity! But do not worry)

—

What else is more important ...?

(To be discussed.)