Double integrals; brief preview

Fact: For all "sufficiently nice" (i.e., Math 3-relevant) functions $g(x,y)$, the expressions

$$\int_a^b \underbrace{\left[\int_c^d g(x,y)\,dy\right]}_{\text{a function } u(x)} dx \quad \text{and}$$

$$\int_c^d \underbrace{\left[\int_a^b g(x,y)\,dx\right]}_{\text{a function } v(y)} dy \quad \text{are } \underline{\text{equal}}.$$

But sometimes, one is easier to compute than the other.

Ex: $$\int_{-1}^{1} \underbrace{\int_3^4 y e^y \sin(xy)\,dy}_{\text{hard to compute}}\,dx$$

$$= \int_3^4 \underbrace{\int_{-1}^{1} y e^y \sin(xy)\,dx}_{=0}\,dy$$

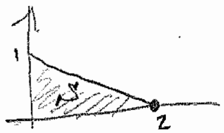
$$\int_{-1}^{1} y e^y \sin(xy)\,dx = e^y\Big[\cos(xy)\Big]_{x=-1}^{x=1} = 0.$$

What more?

→ The (double) integral as limits of sums:
the <u>Riemann</u> integral

↳ $\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}\,dx = ?$ as application

→ Double integrals over other sets. E.g.



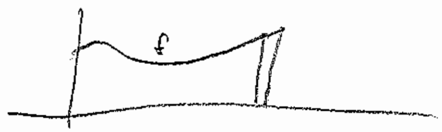
The double integral over S:
x runs from 0 to 2. For each x,
y runs from 0 to $1-\frac{1}{2}x$

$$\int_0^2 \int_0^{1-x/2} \cdots\cdots \, dy\, dx$$

How to change order here?

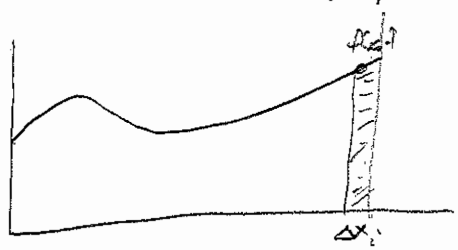
→ The triple integral

The "Math 2" integral (Newton / Leibniz)



$F'(x) = f(x)$.

For the "area under graph" interpretation:



area $\approx f(x_i)\Delta x_i$

Total $\approx \sum f(x_i)\Delta x_i$.

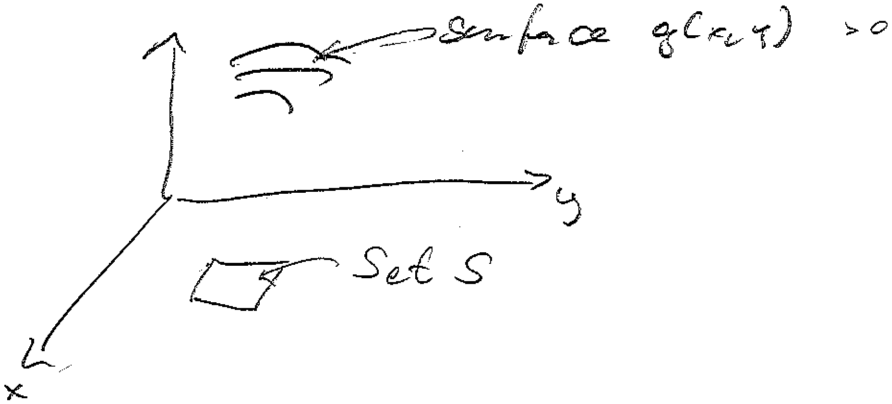
We can <u>define</u> the integral as

$\lim \sum f(x_i)\,\Delta x_i$ where the limit is taken as $\max|\Delta x_i| \to 0$.

Integral defined as long as we get the same number out no matter how we "refine".

This is the Riemann definition of the integral.

The Riemann "double integral":



- Partition S into smaller sets S_i each of area ΔA_i
- For each i: pick a point $(x_i, y_i) \in S_i$
- Approximate volume under graph by
  $g(x_i, y_i)\,\Delta A_i$
- Total volume $\approx \sum_i g(x_i, y_i)\,\Delta A_i$
- Refine into smaller sets; if $\sum g\,\Delta A_i$ converges to the same number no matter how we refine, this is the double integral of g over S.
  $\iint_S g\,dA$
- Fact: this works for all continuous g (and many more!)

Fact:

If S is a rectangle $[a,b] \times [c,d]$,

then
$$\iint_S g(x,y)\,dA = \int_a^b \left[\int_c^d g(x,y)\,dy \right] dx$$
$$= \int_c^d \left[\int_a^b g(x,y)\,dx \right] dy.$$

The two latter are "iterated" integrals:

- Slicing a loaf into small "cubes"

=

- Slicing, calculating area, aggregating

=

- Slicing, ...

Ex: Let (X, Y) be a <u>uniform</u> random vector over $\underline{[1,2] \times [3,4]}$

square, area $= 1$.
probability $=$ area.

Find $E[X\sqrt{Y}]$.

Equals $\displaystyle\iint_{[1,2]\times[3,4]} x\sqrt{y} \cdot 1 \, dA$

pdf

Good thing: Can evaluate as

$$= \int_1^2 \int_3^4 x\sqrt{y} \, dy \, dx$$

x runs from 1 to 2,
and for each x:
y runs from 3 to 4.

or: other way!

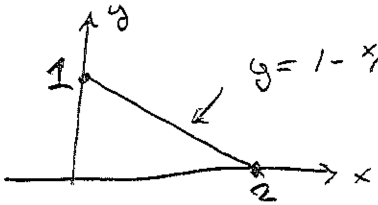
Calculate:

$$\int_1^2 x \left[\frac{2}{3} y^{3/2}\right]_{y=3}^{y=4} dx$$

$$= \int_1^2 x \cdot (16 - 2\sqrt{3}) \, dx = \underline{\underline{3 \cdot (8 - \sqrt{3})}}.$$

Non - rectangles. Triangles first

Let S be the set



and $g(x,y) > 0$.

- Find the volume under the graph over S.

* Solution:

We need to cover all points of S, (and precisely once!)

x runs from 0 to 2

and for each $x \in (0,2)$:

y runs from the x-axis to the line

i.e: y runs from 0 to $1 - x/2$.

$$\int_0^2 \underbrace{\int_0^{1-x/2} g(x,y)\,dy}_{\text{function of only } x}\,dx$$

Alternatively:

y runs from 0 to 1

and for each y,

x runs from 0 to the line

The line: $y = 1 - x/2$ i.e.

$x = 2 - 2y$

$$\int_0^1 \Big(\underbrace{\int_0^{2-2y} g(x,y)\, dx}_{\text{function of } y \text{ only.}} \Big)\, dy$$

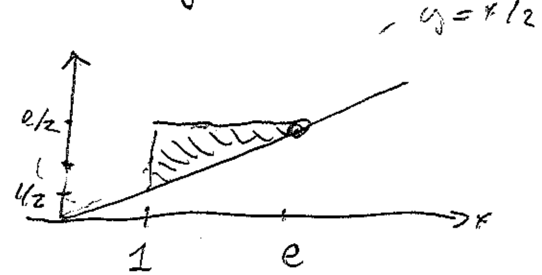
If the innermost integral does not rid you of the variable you integrated up, you have done something <u>very</u> wrong!

Ex: Calculate

$$\int_1^e \int_{x/2}^{e/2} \frac{1}{y^3+xy^2}\, dy\, dx$$

by reversing order.

Set:



y runs from $\frac{1}{2}$ to $\frac{e}{2}$

and for each y:

x runs from 1 to the line $x=2y$

$$\int_{1/2}^{e/2} \int_1^{2y} \frac{1}{y^2} \frac{1}{x+y}\, dx\, dy$$

$$= \int_{1/2}^{e/2} \underbrace{\frac{1}{y^2}\left(\ln(2y+y) - \ln(1+y)\right)}\, dy = \ldots$$

antiderivative: $\ln(y+1) - \ln y + \dfrac{\ln(y+1) - \ln y - \ln 3 - 1}{y}$

(Wolfram Alpha ...)

Worse sets: Careful!

$$\int_{-1}^{1} \int_{x^2/2}^{2x^2} g(x,y)\, dy\, dx$$

x from -1 to 1

y from $x^2/2$ to $2x^2$

So y could be as small as 0
and as large as 2.

y runs from 0 to 2
and for each y:

x from ... what?!?

Small y: 

and then 

Large y: -1 to \, & / to 1

<u>Don't!</u> Calculate this as $dy\,dx$.

(Exam? Don't worry!)

Triple integral ?

Box sets only:

$$\int_a^b \int_c^d \int_p^q g(x,y,z)\, dz\, dy\, dx$$

$$= \int_c^d \int_p^q \int_a^b g\, dx\, dz\, dy \ldots\ldots$$

Finally, if time permits:

Volume under $g = e^{-\frac{1}{2}(x^2+y^2)}$

Set $= \mathbb{R}^2$.

g is a function of only $r = \sqrt{x^2+y^2}$.

cylinder shell: $2\pi r \cdot \Delta r \cdot g$

r from 0 to $+\infty$: $2\pi \int_0^\infty r\, e^{-\frac{1}{2} r^2}\, dr$.

Note: This $= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\, e^{-\frac{y^2}{2}}\, dy$

$$= \left(\int_{-\infty}^{\infty} e^{-x^2/2}\, dx \right)^2$$