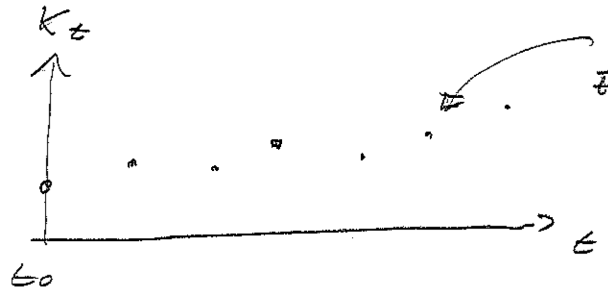
Dynamic optimization

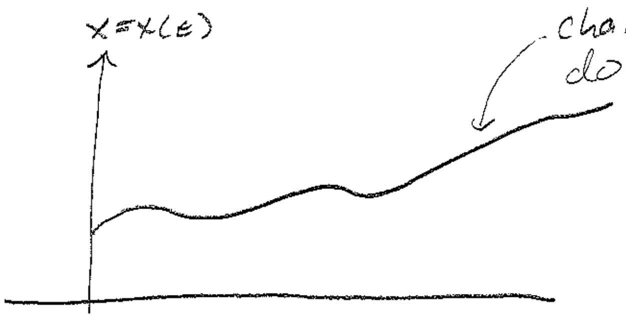
Find the best "path" or function.

Discrete time can be treated as T-variable optimization



Euler eq = FOC for "this".

Continuous time:



changing $x(t_0)$ only does nothing good if objective is a ~~dt~~-integral.

Continuous-time dynamic optimization:

Typical problem: trade-off between consumption now and saving/investing for later.

State $= x(t)$. Today, we can choose $\dot{x}(t)$.

Think: High $\dot{x}$ $\to$ more of x tomorrow, but less to consume today.

Tools on curriculum:

* The Euler equation from calculus of variations (1696–1750s)
  - $\to$ Full control of $\dot{x}$
  - $\to$ FOC leads to 2nd order diff. eq.
* Pontryagin's maximum principle (1956 ff tool of optimal control)
  - $\to$ $\dot{x} = g(t, x, u)$. (control)
    Not necessarily "full control" over $\dot{x}$.
  - $\to$ Shadow price p on state
  - $\to$ Leads to D.E. system $\begin{pmatrix} \dot{x} \\ \dot{p} \end{pmatrix} = \vec{a}(t, x, p)$

Calculus of variations & the Euler equation

Problem: $\max / \min \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t)) \, dt$, $x(t_0) = x_0$, $x(t_1) = x_1$

where t_0, t_1, x_0, x_1 are given, and we maximize over all C^1 functions x starting at $x(t_0) = x_0$, ending at $x(t_1) = x_1$.

(or maybe just piecewise)

Notes on notation:

→ Book uses F for running utility in this chapter, f in the next.

→ F "must" depend on $\dot{x}$, otherwise there is no dynamic trade-off.
Use $\frac{\partial F}{\partial \dot{x}}$ for partial derivative wrt 3^{rd} variable
Alternatively: $\frac{\partial F}{\partial u}$, $(u = \dot{x})$ – or F_3'.

There are tools for cases "$x(t_1)$ free" or "$\geq x_1$", but we cover those under the maximum principle.

The <u>Euler equation</u> for the problem:

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0$$

<u>total</u> derivative, note that $x = x(t)$, $\dot{x} = \dot{x}(t)$.

(The Euler eq. is 2^{nd} order if $F''_{32} \neq 0$:

$$F''_{33}\ddot{x} + F''_{32}\dot{x} + F''_{31} - F'_2 = 0$$

(harder to remember!)).

Fact: Let $F \in C^2$.

Necessary cond's: The Euler eq & $x^*(t_0) = x_0$, $x^*(t_1) = x_1$.

Sufficient: The necessary & in addition

$(x, \dot{x}) \to F$ concave, each $t \in (t_0, t_1)$ for max

$(x, \dot{x}) \to F$ convex ——— , ——— min

Some special cases:

- If F has no $\dot{x}$ - dependence: Not a dynamic problem. max over x.
- If F has no x - dependence:

  $\frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 \Rightarrow \frac{\partial F}{\partial \dot{x}} = C$

  First-order. Gives a 2^{nd} constant of integration D.
- If F has no t - dependence, one can show: $F - \dot{x}\frac{\partial F}{\partial \dot{x}} = C$ as long as $\dot{x} \neq 0$. "first integral".

  Not <u>needed</u> @ exam, could simplify. (1^{st} order.)
- If F neither has t nor x - dependence

  No "x": $\frac{\partial F}{\partial \dot{x}} = C$ only depends on $\dot{x}$ (no "t")

  leads to $\dot{x} = \text{constant}$. $x = x_0 + \frac{x_1 - x_0}{t_1 - t_0}(t - t_0)$

Example: Let $c>0$ constant.

$$\max \int_{2019}^{2021} (x - c\dot{x}^2)\,dt \quad \Big| \quad \begin{array}{l} x(2019) \text{ given} \\ x(2021) \text{ given.} \end{array}$$

Euler eq.: $0 = \underbrace{\frac{\partial}{\partial x}(x - c\dot{x}^2)}_{1} - \underbrace{\frac{d}{dt}\underbrace{\frac{\partial}{\partial \dot{x}}(x - c\dot{x}^2)}_{-2c\dot{x}}}_{-2c\ddot{x}}$

So
$$2c\ddot{x} + 1 = 0$$
$$2c\dot{x} + t = A$$
$$2cx + \tfrac{1}{2}t^2 = At + B$$

Fit A and B to $x(2019)$ and $x(2021)$.

Arguably "better" exposition:
$$2c\dot{x} + (t - 2019) = P$$
$$2c\,(x - x(2019)) + \tfrac{1}{2}(t-2019)^2 = P\cdot(t-2019)$$
fits the "2019" condition

P such that
$$2c\,(x(2021) - 2019) + \tfrac{1}{2}\cdot 2^2 = P\cdot 2$$
gives $P = 1 + (x(2021) - x(2019))\cdot c$

Now $x(t) = x(2019) + \left[\frac{x(2021) - x(2019)}{2} + \frac{1}{2c}\right](t - 2019)$
$$- \frac{1}{4c}(t-2019)^2$$

Example (bode): convex in $(x,\dot{x})$

$$\min \int_0^T \left(x^2 + c\dot{x}^2\right) dt \,, \qquad \begin{array}{l} c>0 \text{ constant} \\ x(0) = x_0 \text{ given} \\ x(T) = 0. \end{array}$$

notation: means "$(x(t))^2 + c(\dot{x}(t))^2$".

* Euler eq.:

$$0 = \underbrace{2x}_{\frac{\partial F}{\partial x}} - \frac{d}{dt}\underbrace{(2c\dot{x})}_{\frac{\partial F}{\partial \dot{x}}} = 2x - 2c\ddot{x}$$

yields $\ddot{x} - \frac{1}{c}x = 0$. $r^2 - \frac{1}{c} = 0$. Put $g = c^{-1/2}$

General sol'n $A e^{gt} + B e^{-gt}$.

Fit constants: $A + B = x_0$

$A e^{gT} + B e^{-gT} = 0$ [...]

yields $$x^*(t) = x_0 \frac{e^{g(T-t)} - e^{-g(T-t)}}{e^{gT} - e^{-gT}}$$

(* "Alt.:" By first integral: $G = F - \dot{x}\frac{\partial F}{\partial \dot{x}}$

$$= x^2 + c\dot{x}^2 - \dot{x}\cdot 2c\dot{x}$$

$$= x^2 - c\dot{x}^2$$

$$\dot{x} = c^{-1/2}\sqrt{x^2 - G}$$ not easy!

Check Wolfram Alpha for the solution, try to reverse-engineer by differentiating and integrate with the steps done backwards; still not easy!)

Example:

The arc length of a curve: $\int \sqrt{(dt)^2 + (dx)^2}$ (by Pythagoras) $= \int \sqrt{1 + \left(\frac{dx}{dt}\right)^2}\, dt$.

min/max $\displaystyle\int_0^T \sqrt{1 + \dot{x}^2}\, dt$, $\quad x(0) = x_0$, $x(T) = x_T$

Euler eq:

$$0 = \overset{\partial F/\partial x}{\overset{\downarrow}{0}} - \frac{d}{dt} \frac{\partial}{\partial \dot{x}} \sqrt{1 + \dot{x}^2}$$

So $\quad \dfrac{\partial}{\partial \dot{x}} \sqrt{1 + \dot{x}^2} = C$

$$\sqrt{1 + \dot{x}^2} = C\dot{x} + D$$

$\dot{x}$ depends only on constants $\rightsquigarrow$ $\dot{x}$ is a constant.

x = straight line. Surprised?

Application: Ramsey's consumption/savings problem

$$\underset{\text{output}}{Y} = f(\underset{\text{capital}}{k}) \qquad (f' > 0 \geq f'')$$

$$Y(t) = \overset{\text{consumption}}{c(t)} + \overset{\text{investment}}{\dot{k}(t)} \qquad c = f(k) - \dot{k}$$

- utility $U(c)$ from consumption, $U' > 0 > U''$. Problem:

$$\max \int_0^T \underbrace{U(f(k) - \dot{k})}_{\text{concave in } (k,\dot{k}) \text{ (why?)}} \overset{\text{discounted, } r \geq 0}{e^{-rt}} dt \qquad \text{s.t.} \quad \left.\begin{array}{l} k(0) = k_0 \\ k(T) = k_T \end{array}\right\} \text{given.}$$

State $= k$.

Euler Eq.:

$$0 = \overbrace{U'(c) f'(k) e^{-rt}}^{\frac{\partial F}{\partial k}} - \frac{d}{dt}\left(U'(c) \cdot (-1) e^{-rt}\right)$$

$$= U'(c) f'(k) e^{-rt} + U''(c) \dot{c} e^{-rt} - r U'(c) e^{-rt}$$

$$0 = U'(c) \cdot (f'(k) - r) + \dot{c}\, U''(c).$$

often written:

$$\frac{\dot{c}}{c} = (r - f'(k)) \Big/ \underbrace{El_c U'(c)}_{\text{elast. of mg utility, } < 0.}$$

Consumption <u>increases</u> as long as $f'(k) - r > 0$,
i.e. mg. prod. of capital > discount rate.

Can insert $\dot{c} = f'(k)\dot{k} - \ddot{k}$ .....

If time: <u>Why</u> the Euler eq. ?

Consider a path $x = x(t)$. Modify it: $x + \overset{\text{constant}}{\alpha}\,\underset{\text{function}}{\mu}$.

(the "variation" in "calculus of variations")

Get:

$$\int_{t_0}^{t_1} F(t,\, x+\alpha\mu,\, \dot{x}+\alpha\dot{\mu})\, dt$$

If $x = x^*$ is optimal, the "best" variation is 0.

So take $\frac{d}{d\alpha}$ and insert $\alpha = 0$; that must yield 0.

$$\int_{t_0}^{t_1} \left(\frac{\partial F}{\partial x}\mu + \frac{\partial F}{\partial \dot{x}}\dot{\mu} \right) dt = 0 \quad \text{when } x = x^*.$$

Trick: integrate by parts to "turn $\dot{\mu}$ into μ":

$$\int_{t_0}^{t} \frac{\partial F}{\partial \dot{x}}\dot{\mu}\, dt = \Big|_{t_0}^{t_1} \frac{\partial F}{\partial \dot{x}}\mu - \int_{t_0}^{t_1} \mu \frac{d}{dt}\frac{\partial F}{\partial \dot{x}}\, dt$$

Since $x^* + \alpha\mu = \begin{cases} x_0 & \text{at } t_0 \\ x_1 & \text{at } t_1, \end{cases}$
$\mu(t_0) = \mu(t_1) = 0$.

So

$$\int_{t_0}^{t_1} \mu(t) \left(\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} \right) dt = 0 \quad \text{when } x = x^*,$$

to hold for <u>all</u> μ with $\mu(t_0) = \mu(t_1) = 0$.

But: Make sure that μ has same sign as $\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}}$

Then the latter must be 0 to get 0.