Difference equations

→ a.k.a.: "recurrence relations", "iterative mappings"...

1st order:

only last step.

$$x_{t+1} = f(t, x_t)$$

(This is the common form, although "difference eq." suggests the equivalent

$$\underbrace{x_{t+1} - x_t}_{\text{difference, cf. the differential eq. formulation}} = g(t, x_t) \quad \{ g = -x_t + f$$
)

* As long as f is defined everywhere:
Existence & uniqueness granted (with initial state specif. cf. differential eq.s)
No "explosions" in finite time.

* Ex.: $\begin{cases} x_{t+1} = a\,x_t, \\ x_0 \text{ given} \end{cases}$

$\Rightarrow \quad x_t = a^t x_0$ fits

(and is the unique solution)

Behaviour of

$$x_{t+1} = a x_t,$$

x_0 given

→ $a x_0 = 0$ ⇒ $x_t = 0 \ \forall t$. <u>Otherwise:</u>

→ If $a > 1$: $x_t \to x_0 \cdot (+\infty)$

→ If $a = 1$: $x_t = x_0 \quad \forall t$

→ If $a \in (0,1)$: $x_t \to 0$

- monotonically!

→ If $a \in (-1,0)$: Dampened oscillations.

$x_t \to 0$, but alternating sign.

→ If $a = -1$: Oscillations: $|x_t|$ const.,

$x_t = (-1)^t x_0$

→ If $a < -1$: Explosive oscillations

Stable: ⇔ $|a| \leq 1$

Globally asymptotically stable: $|a| < 1$

Notice different threshold values than for the differential eq. $\dot{x} = ax$.

The linear difference eq.

→ constant coefficients,

$$x_{t+1} = a x_t + b \quad \Big| \quad x_0 \text{ given}$$

$$\begin{aligned}
x_1 &= a x_0 + b \\
x_2 &= a(a x_0 + b) + b \\
&= a^2 x_0 + (a+1) b \\
x_3 &= a(a^2 x_0 + (a+1) b) + b \\
&= a^3 x_0 + \underbrace{(a^2 + a + 1)}_{\text{geometric series.}} b
\end{aligned}$$

The solution can be proven by <u>induction</u>:

Claim: $$x_t = a^t x_0 + b \cdot \begin{cases} \dfrac{1-a^t}{1-a} & , a \neq 1 \\ t & , a = 1 \end{cases}$$

* True for $t = 0$.
* Induction step for the "harder" case $a \neq 1$:
  * Suppose true for some $t = T$
  * Then $$\begin{aligned} x_{T+1} &= a x_T + b \\ &= a\left(a^T x_0 + b \frac{1-a^T}{1-a}\right) + b \\ &= a^{T+1} x_0 + \left(a \cdot \frac{1-a^T}{1-a} + 1\right) b \end{aligned}$$

$$a \frac{1-a^T}{1-a} + \frac{1-a}{1-a} = \frac{1 + a - a - a^T \cdot a}{1-a}, \ \ldots$$

Example: Loan at interest rate $r > 0$
constant downpayment c
(both per period)

$$x_{t+1} = (1+r)\, x_t - c \quad , \quad x_0 > 0$$

$$x_t = (1+r)^t \left[x_0 - \frac{c}{r} \right] + \frac{c}{r}$$

→ decreases if $c > x_0 r$ (surprise!)

→ Q: when is the loan fully repaid?

A: x_t will only <u>hit</u> 0 in special cases, so we solve for first time T_0 for which $x_{T_0} < 0$:

$$-\frac{c}{r} > (1+r)^T \left[x_0 - \frac{c}{r} \right]$$

$$(1+r)^T > \frac{c/r}{x_0 - c/r} = \frac{c}{r x_0 - c}$$

$$T > \frac{1}{1+r} \ln \frac{c}{r x_0 - c}$$

T_0 : first T for which this is true.

Notice also:

Just like for linear differential equations, we can

- look for a solution of the corresponding homogeneous

$$x_{t+1} = a x_t \quad : \quad u_t = a^t$$

- look for a particular solution of

$$x_{t+1} = a x_t + b$$

$\Rightarrow$ stationary state when

$x_{t+1} = x_t = u^*$: works if $a \neq 1$,

then $u^* = \dfrac{b}{1-a}$.

General solution, $a \neq 1$:

$$C a^t + \frac{b}{1-a}$$

+ $C \Leftrightarrow x_0$? Put $t=0$:

$$x_0 = C + \frac{b}{1-a}$$

We have an analogue to the integrating factor approach: Put

$$y_t = a^{-t} x_t$$

$$y_{t+1} = a^{-(t+1)} x_{t+1} = a^{-(t+1)} (a x_t + b)$$

$$= y_t + b a^{-(t+1)}$$

So: $$y_t = y_0 + b \underbrace{(a^{-1} + a^{-2} + \dots + a^{-t})}_{= \frac{1 - a^{-t}}{a - 1}} \quad \text{if } a \neq 1$$

$$x_t = a^t y_t$$

$$= a^t y_0 + b \frac{a^t - 1}{a - 1}$$

$$= a^t x_0 + b \frac{1 - a^t}{1 - a}$$

The variable coeff. case

We will need a piece of notation:

just like $\sum_{i=1}^{I} c_i$ denotes $c_1 + \ldots + c_I$,

we let $\prod_{i=1}^{I} c_i$ denote $c_1 \cdot \ldots \cdot c_I$

Note: " $\sum_{2}^{1} c_i$ " – the empty sum – is $\underline{\underline{0}}$.

" $\prod_{2}^{1} c_i$ " – the empty product – is $\underline{\underline{1}}$

Consider

$$x_{t+1} = a_t x_t + b_t$$

given functions of t

$$x_1 = a_0 x_0 + b_0$$

$$x_2 = a_1 a_0 x_0 + a_1 b_0 + b_1$$

$$x_3 = a_2 a_1 a_0 x_0 + a_2 a_1 b_0 + a_2 b_1 + b_2$$

$\vdots$

Form:

$$X_t = X_0 \prod_{s=0}^{t-1} a_s + \sum_{s=0}^{t-1} \left(\prod_{r=s+1}^{t-1} a_r \right) b_s$$

Proof (induction):

- OK for $t=0$:

  X_0 (empty product)

  $+$ empty sum $(\quad)$

- Suppose true for $t=T$. Then

$$X_{T+1} = a_T X_0 \prod_{s=0}^{T-1} a_s + a_T \sum_{s=0}^{T-1} \left[\prod_{r=s+1}^{T-1} a_r \right] b_s + b_T$$

$$= X_0 \prod_{s=0}^{T} a_s + \sum_{s=0}^{T-1} \left[\prod_{r=s+1}^{T} a_r \right] b_s + b_T$$

Now $$b_T = \underbrace{\sum_{s=T}^{T}}_{\text{one term}} \Big(\underbrace{\prod_{r=s+1}^{T}}_{\text{empty product}} a_r \Big) b_T$$

and joining the sums, we get the result.

Ex:

$$x_{t+1} = \frac{1}{t+1} x_t + \overset{\text{constant}}{b}$$

$$\prod_{s=0}^{t-1} a_s = 1 \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \ldots \cdot \frac{1}{t} = \frac{1}{t!}$$

$$x_t = \frac{x_0}{t!} + \sum_{s=0}^{t-1} \underbrace{\left(\underset{a_{s+1}}{\frac{1}{s+2}} \cdot \ldots \cdot \underset{a_{t-1}}{\frac{1}{t}} \right)}_{\text{not much to do about this.}} b$$

$$\frac{1}{2 \cdot \ldots \cdot t} + \frac{1}{3 \cdot \ldots \cdot t} + \ldots + \frac{1}{t}$$

$$= \frac{1 + 2 + 3! + \ldots + (t-1)!}{t!}$$

$$x_t = \frac{1}{t!} \left(x_0 + b \left(1 + 2! + \ldots + (t-1)! \right) \right)$$

(note: empty sum for t=0)

Second order

$$F(t, x_t, x_{t+1}, x_{t+2}) = 0$$

This course:

$b \neq 0$ otherwise 1st order

$$x_{t+2} + a x_{t+1} + b x_t = c_t \qquad (*)$$

constant

$c = c_t$

As for differential eq's:

→ Consider the homogeneous case $c_t = 0$ (H)

→ Find two independent solutions u, v

→ "indep": $\begin{vmatrix} u_0 & v_0 \\ u_1 & v_1 \end{vmatrix} \neq 0$

→ General solution of (H):

$$Ku + Lv$$

→ Find a particular solution u^* of $(*)$

→ $(*)$ has general solution

$$Ku + Lv + u^*$$

Method:

Try m^t in (**):

$$m^{t+2} + a m^{t+1} + b m^t = 0$$

Char. eq.: $m^2 + am + b = 0$

Roots: $m = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b}$

- Case $m_1 \neq m_2$ real:

$$K_1 m_1^t + K_2 m_2^t$$

- Case double root $m = -\frac{a}{2}$ $\left(\Leftrightarrow b = \left(\frac{a}{2}\right)^2\right)$

$$(K_0 + K_1 t)\left(-\frac{a}{2}\right)^t$$

- Case of no real roots, $b > \left(\frac{a}{2}\right)^2$

$$x_t = b^{t/2}\left(K_1 \cos(\theta t) + K_2 \sin(\theta t)\right)$$

where $\theta \in [0, \pi]$ s.t. $\cos\theta = -\frac{a}{2\sqrt{b}}$

(alternative:

$$K b^{t/2} \cos(\omega + \theta t)$$

)

Example: the Fibonacci numbers

$$x_{t+2} = x_{t+1} + x_t \quad , \quad x_0 = 0, \ x_1 = 1$$

$$a = b = -1.$$

$$m = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{1+4} = \tfrac{1}{2}\left(1 \pm \sqrt{5}\right)$$

General solution:

$$K_1 \left(\frac{1+\sqrt{5}}{2}\right)^t + K_2 \left(\frac{1-\sqrt{5}}{2}\right)^t$$

Particular solution s.t. $x_0 = 0$, $x_1 = 1$:

$$\begin{aligned} K_1 + K_2 &= 0 \quad \Big| \ \frac{\sqrt{5}-1}{2} \\ K_1 \cdot \frac{1+\sqrt{5}}{2} + K_2 \cdot \frac{1-\sqrt{5}}{2} &= 1 \end{aligned}$$

$$K_1 \sqrt{5} = 1$$

$$K_1 = -K_2 = \tfrac{1}{5}\sqrt{5}$$

$$x_t = \tfrac{1}{5}\sqrt{5}\left[\left(\frac{1+\sqrt{5}}{2}\right)^t - \left(\frac{1-\sqrt{5}}{2}\right)^t\right]$$

Note: all x_t are natural numbers, despite the $\sqrt{5}$'s and fractions!

The nonhomogeneous case

→ There is a general formula. (End of 11.3). Ugly!

→ Will only cover $c_t = d^t$

- try $A d^t$

$$A\left(d^{t+2} + a d^{t+1} + b d^t\right) = d^t$$

$$A = \frac{1}{d^2 + ad + b}$$ if d not a root of the char. eq.

- If $d^2 + ad + b = 0$:

try $A \cdot t d^t$

$$A\left[(t+2) d^{t+2} + (t+1) a d^{t+1} + b t d^t\right] = d^t$$

The $t d^t$ terms vanish

$$A \cdot \left[2d^2 + ad + b\right] = 1$$

Stability: More precisely, global asymptotic stability:

Q: When will the solution of (H) $\longrightarrow 0$ as $t \to \infty$?

A: Two equivalent conditions:

(i) $|a| < 1 + b$ & $b < 1$

(ii) If roots are real: $|m_i| < 1$, $i = 1, 2$

double root $|m| < 1$

no real roots:

$$\sqrt{\left(\frac{a}{2}\right)^2 + \sqrt{b - \left(\frac{a}{2}\right)^2}^{\,2}} < 1$$

i.e. $b < 1$

Non-ex: Fibonacci: $\frac{1+\sqrt{5}}{2} > 1$.

Ex:

$$X_{t+2} - \frac{3}{2} X_{t+1} + \frac{3}{4} X_t = C_t$$

$$|a| = \frac{3}{2} = \frac{6}{4} \qquad 1 + b = \frac{7}{4} \qquad b < 1.$$