ECON4140 2019 (draft): diff.eq. (systems).

Draft $\rightsquigarrow$ could be revised, especially when it comes to what is exam relevant.

FMEA 6.1–6.3 (Norw.: MA2 2.1–2.3) You must know that if you are given two non-proportional solutions u_1 and u_2 of a *homogeneous linear 2nd order* diff.eq., you obtain the general solution as an arbitrary linear combination $C_1 u_1 + C_2 u_2$. (FMEA thm 6.2.1, MA2 setning 2.2.1 jf. 2.2.2). In the non-homoegeneous case, you need the general solution of the corresponding homogeneous, plus a particular solution u^* of the inhomogeneous.

- Non-constant coefficients: not in itself curriculum: you won't get that equation, and "solve!". But given how to find u_1, u_2, u^*, you shall be able to apply the general theory to come up with the answer $C_1 u_1 + C_2 u_2 + u^*$.
- In particular, this goes for Euler's differential equation (a subsection in FMEA; MA2 avsnitt 2.5). See the seminar problem and the remark given.

Constant coefficients (but variable right-hand side $f(t)$): You shall be able to solve the homogeneous completely, and the types of RHS's given. In particular: if the RHS is t^m, then you might need lower-order terms than m; and, if the RHS is $e^{\delta t}\sin(qt)$, you need $e^{\delta t}\left(K\cos(qt) + L\sin(qt)\right)$, etc. Note also what happens if $f(t)$ is a particular solution of the corresponding homogeneous equation.

Constant coeff's only: Stability of linear 2nd order diff.eq's (FMEA section 6.4, MA2 avsnitt 2.6) – and systems in $\mathbb{R}^2$ (FMEA sections 6.6–6.7, MA2 avsnitt 2.7–2.8). For 2nd order diff.eq's $\ddot{x} + a\dot{x} + bx = 0$: [1]

- Globally asymptotically stable $\iff a > 0$ & $b > 0$.
- If $a < 0$ and/or $b < 0$: unstable.
- If $b > 0 = a$: Undampened oscillations $A\cos(t\sqrt{b}) + B\sin(t\sqrt{b})$. Stable, but not asymptotically so (sometimes referred to as «neutrally stable») in the literature.
- If $b = 0 = a$, i.e., the equation $\ddot{x} = 0$: The affine function $C + Dt$, unstable.

Systems $\binom{\dot{x}}{\dot{y}} = \mathbf{A}\binom{x}{y}$ in $\mathbb{R}^2$ (see however handwritten attachment):

- Reduced to 2nd order diff.eq.'s, with characteristic roots = the eigenvalues of $\mathbf{A}$, that is: $\lambda_{\pm} = \frac{\mathtt{tr}\,\mathbf{A}}{2} \pm \sqrt{(\frac{\mathtt{tr}\,\mathbf{A}}{2})^2 - \mathtt{det}\,\mathbf{A}}$. As with 2nd orders: if these are not real numbers, you put $\alpha = \frac{\mathtt{tr}\,\mathbf{A}}{2}$ and $\beta = \sqrt{\mathtt{det}\,\mathbf{A} - (\frac{\mathtt{tr}\,\mathbf{A}}{2})^2}$ and use the formula. (See remark at the end concerning complex numbers.)

[1] Norwegian: Enkelte gamle utgaver av MA2 har en trykkfeil om det følgende. 2013-versjonen er riktig.

- In particular, you are expected to tell whether an *unstable* system is a *saddle point:* when the eigenvalues are real and of opposite sign ($\Leftrightarrow \texttt{det}\,\mathbf{A} < 0$). For saddle points you shall know that the *convergent* particular solutions have the slope of the eigenvector $\mathbf{v}$ associated to the *negative* eigenvalue. (Slope $y(t)/x(t) = v_2/v_1$ then.)

The "simplest" linear systems $\dot{\vec{x}} = \mathbf{A}\vec{x}$ in $\mathbb{R}^n$: If $\mathbf{A}$ (assumed constant!) has n *distinct* real eigenvalues $\lambda_1, \ldots, \lambda_n$ with eigenvectors $\mathbf{v}^{(1)}, \ldots, \mathbf{v}^{(n)}$ then: $C_1\mathbf{v}^{(1)}e^{\lambda_1 t} + \cdots + C_n\mathbf{v}^{(n)}e^{\lambda_n t}$. These C_i can be fit to any given initial state; if $\vec{x}(0)$ is given, then $\vec{C} = \mathbf{V}^{-1}\vec{x}(0)$, where $\mathbf{V}$ has the eigenvalues as columns.
(Also true, but not curriculum relevant: replace "real" by "complex".)

Nonlinear systems. (No global asymptotic stability, so NOT FMEA Theorem 6.8.2.)

- Phase plane analysis: exam ambition is to «sketch» a phase diagram, so that things look qualitiatively right: trajectories should be vertical/horizontal across the respective nullclines, up/down and left/right directions should be correct).

- Classification of equilibrium points through eigenvalues of the Jacobian. Note the cases where the test is inconclusive: if the largest eigenvalue is zero, we cannot tell if stable or not; if $\lambda_2 > \lambda_1 = 0$ it is unstable, but we cannot tell whether it is a saddle point.
  - In particular, concerning saddle points: The particular solutions that converge to the saddle point, will *in the limit* converge like the eigenvector $\mathbf{v}$ corresponding to the negative eigenvalue. More precisely: If the saddle point is (x_*, y_*), then $\lim_{t\to+\infty} \frac{y(t)-y_*}{x(t)-x_*} = v_2/v_1$ (except slope $\to$ vertical if $v_1 = 0$).

- *Not* curriculum: Lyapunov functions and Olech's theorem (part of FMEA sec. 6.8; MA2 avsnitt 2.12). One seminar problem is taken from there, but that problem only asks you for *local* classification, which *is* curriculum.

Complex numbers: you can do without! At the exam you need nothing. In teaching, you will have to deal with the term «real part». The real part of a real number is the number itself, but if the formula for a quadratic yields, e.g., $r = 7 \pm \sqrt{-4}$, then the real part is 7. (Regardless of whether «±» is + or −.) Take note that if the quadratic function has no zeroes, then this real part is the stationary point.
Similar goes for a 2×2 matrix $\mathbf{A}$ with characteristic equation $\lambda^2 - \lambda\,\texttt{tr}\,\mathbf{A} + \texttt{det}\,\mathbf{A} = 0$: Formula yields $\frac{\texttt{tr}\,\mathbf{A}}{2} \pm \sqrt{(\frac{\texttt{tr}\,\mathbf{A}}{2})^2 - \texttt{det}\,\mathbf{A}}$. Real part $= \lambda$ if real; otherwise, $\frac{\texttt{tr}\,\mathbf{A}}{2}$.

"Non-borderline" cases $\quad \max_k \operatorname{Re}\lambda_k \neq 0$

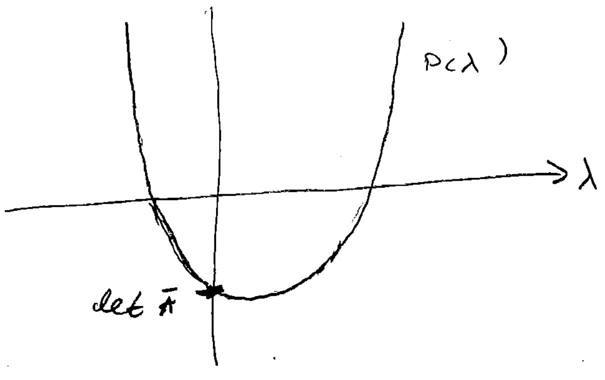
⊛ Generally, $\mathbb{R}^n$

$$\max_k \operatorname{Re}\lambda_k < 0 \Rightarrow \begin{cases} \text{local asympt. stability} \\ \text{global}\overbrace{\text{ if}}^{\text{asymp. st.}}\text{ the system is linear.} \end{cases}$$

$$\max_k \operatorname{Re}\lambda_k > 0 \Rightarrow \text{unstable}$$

⊛ $\mathbb{R}^2$, $\quad \boxed{\underbrace{\lambda^2 - \lambda \operatorname{tr}\bar{A} + \det\bar{A}}_{p(\lambda)} = 0}$

• Case $\det\bar{A} < 0$: $\quad p(0) < 0$



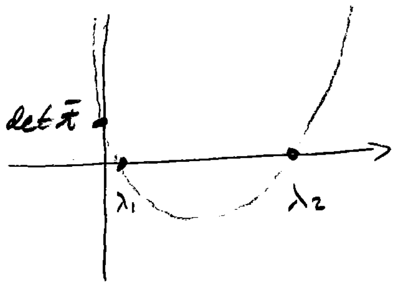
Opposite-sign real eigenvalues.
(Unstable) saddle point.

$\mathbb{R}^2$ cont'd,

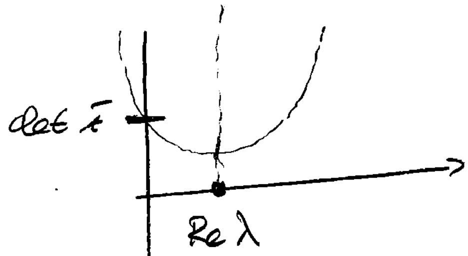
cases with $\det \bar{A} > 0$:

$$p(0) = \det \bar{A} > 0$$
$$p'(0) = -\operatorname{tr} \bar{A}$$

- $\operatorname{tr} \bar{A} > 0$:

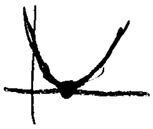


or



→ Both eigenvalues real & positive. Unstable. ("Source")

→ Also valid for the double-eigenvalue case $\lambda_1 = \lambda_2 > 0$

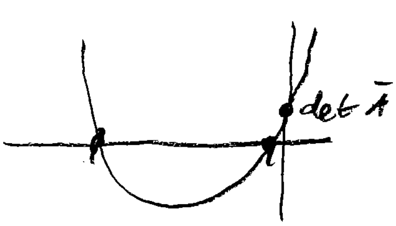


→ No real λ.

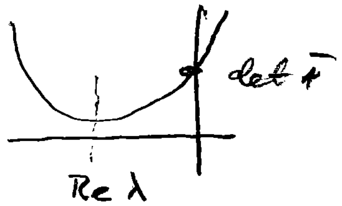
→ $\operatorname{Re} \lambda$ = the symmetry axis = the minimum point for $p(\lambda)$ $= \frac{\operatorname{tr} \bar{A}}{2}$.

→ $\operatorname{Re} \lambda > 0$ ⇒ unstable, outwards spiraling

- $\operatorname{tr} \bar{A} < 0$: Mirror about vertical axis



or

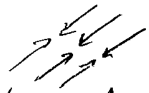


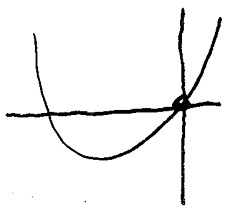
Stable. ("Sink".)
Also valid for $\lambda_1 = \lambda_2 < 0$

No real λ.
$\operatorname{Re} \lambda < 0$.
Stable, spiral inwards

For the following cases, we can classify a <u>linear</u> system $\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \bar{A}\begin{pmatrix}x\\ y\end{pmatrix}$ <u>completely</u>, but the Jacobian $\bar{A} = J(x^*, y^*)$ of a <u>nonlinear</u> system can <u>at most</u> give partial information

- Case $\lambda_1 < \lambda_2 = 0 = \det \bar{A}$ ($\lambda_1 = \operatorname{tr}\bar{A}$)

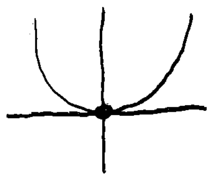




· Linear system: <u>stable</u>, but <u>not</u> asymptotically

· Nonlinear: <u>cannot decide stability</u>, but <u>either sink or saddle</u>

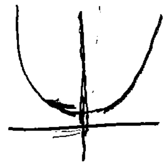
- Cases $\lambda_1 = \lambda_2 = 0$ $= \det \bar{A} = \operatorname{tr}\bar{A}$



Linear: <u>unstable</u> ("C + Dt")

Nonlinear: could be <u>anything</u>!

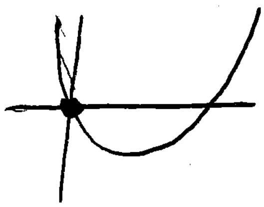
$\det \bar{A} > 0 = \operatorname{tr}\bar{A}$, $\operatorname{Re}\lambda = 0$, no <u>real</u> λ



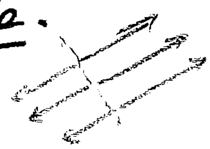
Linear: <u>ellipses</u> (Stable, but not asymptotically)

Nonlinear: <u>cannot decide stability</u>, but: orbiting <u>spirals</u> or <u>ellipses</u>

- Case $0 = \lambda_1 < \lambda_2 = \operatorname{tr}\bar{A}$, $0 = \det \bar{A}$



Linear: <u>Unstable</u>.



Nonlinear: cannot tell source/saddle