# Dynamic optimization: Discrete time

Problem:

$$\max_{u_s, \dots, u_T \in U} / \min \sum_{t=s}^{T} f(t, x_t, u_t)$$

"a payoff at time $t$"

where $x_{t+1} = g(t, x_t, u_t)$.

"max" from now on

I.e.: we choose $u_t \in U$ (fixed set), controlling payoff "now" and the dynamics (hence future payoff)

Example: consumption/investment trade-off.

<u>Method</u>: solve recursively from time $T$

- at $T$: $\max_{u \in U} f(T, x, u)$ yields an optimal $u_T^* = u_T^*(x_T)$ — no future to consider!

  "feedback form" - ok in this course!

- Once time $t$ has been solved, with <u>optimal</u> payoff $J_t(x_t)$:

$$J_{t-1}(x) = \max_{u \in U} \Big\{ \underbrace{f(t-1, x, u)}_{\text{(payoff today)}} + \underbrace{J_t}_{\text{value tomorrow of}} \big( \underbrace{g(t-1, x, u)}_{\text{tomorrow's state}} \big) \Big\}$$

This is the <u>dynamic programming equation</u> - your choice today is based on your choosing optimally from tomorrow on. (Compare: Nash equilibrium "with yourself".)

Ex:

$$\max_{u_t \in [-1,1]} \sum_{t=0}^{4} (x_t + u_t) \quad \text{when} \quad x_{t+1} = x_t - u_t^2$$

$x_0$ given.

time $t=4$: $\max_{u \in [-1,1]} x_4 + u = \underbrace{x_4 + 1}_{J_4(x_4)}$ with $\underline{u_4^* = 1}$.

time $t=3$: $\max_{u \in [-1,1]} \left\{ x_3 + u + J_4 (\underbrace{x_3 - u^2}_{x_4}) \right\}$

tomorrow's state if we use $u$ today

$x_4 \; +1$

$$= x_3 + \max_{u \in [-1,1]} \left\{ u + \overbrace{(x_3 - u^2 + 1)} \right\}$$

Concave wrt $u$; max for $\underline{\underline{u_3^* = \tfrac{1}{2}}}$, value

$$J_3(x_3) = 2x_3 + 1 + \tfrac{1}{2} - \tfrac{1}{4} = \underline{\underline{2x_3 + \tfrac{5}{4}}}$$

$\tfrac{1}{2} \in [-1, 1]$

time $t=2$:

$$J_2(x_2) = \max_{u\in[-1,1]} \{ x_2+u + \underbrace{2(x_2-u^2) + \tfrac{5}{4}}_{J_3(x_3)} \}$$

max for $\underline{\underline{u_2^* = \tfrac{1}{4}}}$, value:

$$J_2(x_2) = 3x_2 + \frac{5}{4} + \frac{1}{4} - 2\cdot\frac{1}{16}$$

$$= \underline{\underline{3x_2 + \frac{11}{8}}}$$

time $t=1$:

$$J_1(x_1) = \max_{u\in[-1,1]} \{ x_1+u + 3(x_1-u^2) + \frac{11}{8} \}$$

max for $\underline{\underline{u_1^* = \tfrac{1}{6}}}$

$$= 4x_1 + \frac{11}{8} + \frac{1}{6} - \frac{3}{36}$$

$$= \underline{\underline{4x_1 + \frac{35}{12}}}$$

time $t=0$:

$$J_0(x_0) = \max_{u\in[-1,1]} \{ x_0+u + 4(x_0-u^2) + \frac{35}{12} \}$$

$$= 5x_0 + \frac{35}{12} + \frac{1}{8} - \frac{4}{64} \quad \text{with } \underline{\underline{u_0^* = \tfrac{1}{8}}}$$

$$= \underline{\underline{5x_0 + \frac{143}{48}}}$$

Done!

But what if the horizon is a general $T$?

Ex:
$$\max_{u \in [-1,1]} \sum_{t=0}^{T} (x_t + u_t) \quad \text{s.t.} \quad x_{t+1} = x_t - u_t^2$$
$x_0$ given.

( "$T$" in place of "$4$").

Copying the previous calculations, we get

$$J_T(x_T) = x_T + 1 \quad \text{with } u_T^* = 1$$
$$J_{T-1}(x_{T-1}) = 2x_{T-1} + \frac{5}{4} \quad \text{with } u_{T-1}^* = \frac{1}{2}$$
$$J_{T-2}(x_{T-2}) = 3x_{T-2} + \frac{11}{8} \quad \text{with } u_{T-2}^* = \frac{1}{4}$$
$$J_{T-3}(x_{T-3}) = 4x_{T-3} + \frac{35}{12} \quad \text{with } u_{T-3}^* = \frac{1}{6}$$

We can guess a pattern:
$$J_{T-k}(x) = (k+1)x + b_k$$

(... we can maybe guess more, but let us stick to this).

True for $k=0$. Use induction.
(Reverse induction: $T-s$, $s = 0, 1, 2, ..$)

Suppose for $k = K$,

$$J_{T-K}(x) = (K+1)\,x + b_K$$

Then for $K+1$:

$$J_{T-(K+1)}(x) = \max_{u \in [-1,1]} \left\{ x + u + (K+1)(x-u^2) + b_k \right\}$$

$$= (K+2)\,x + b_k + \max_{u \in [-1,1]} \left\{ u - (k+1)u^2 \right\}$$

... already here the form is proven! <u>This</u> maximization - and maximum! - is independent of $x$, and we will have

$$b_{k+1} = b_k + \underbrace{\max_{u \in [-1,1]} \left\{ u - (k+1)\,u^2 \right\}}$$

max for $u^*_{T-(k+1)} = \frac{1}{2(k+1)}$

(the max of the parabola - must check the $\in [-1,1]$ !)

$$b_{k+1} = b_K + \frac{1}{2(K+1)} - \frac{(K+1)}{4(K+1)^2}$$

$$= b_k + \frac{1}{4(K+1)}$$

Difference eq. for $b_k$ (with $b_0 = 1$)

Note: Sometimes, this is the level of ambition for the problem. You could be asked

(a) Calculate the value at times $t = T, T-1, T-2$ and $T-3$

(b) Establish a form for the value function.

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or:

(b) show that the value function $J_t(x)$ can be written on the form

$$A_t x_t + B_t,$$

find $A_t$ and deduce a difference eq. for $B_t$.

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Note: sometimes, one (or more) $f$ is specified separately — usually the last one ("scrap value")

Ex: $$\max_{u_t \in \mathbb{R}} \Big\{ \sum_{t=0}^{T-1} (-u_t)^2 + \underbrace{x_T}_{f(T,\, x_T,\, u_T)} \Big\} \qquad x_{t+1} = \max\{x_t \cdot (1+u_t-u_t^2),\ 0\}$$

(ensuring $x_t \geq 0$)

$J_T(x_T) = x_T$, $u_T^*$ <u>arbitrary</u>

$J_{T-1}(x_{T-1}) = \max\left(-u^2 + x_{T-1}(1+u-u^2)\right)$

$x_T \geq 0$ so this is concave in $u$,

max for $u_{T-1}^* = \dfrac{x_{T-1}}{2(1+x_{T-1})}$

(Problem too ugly. Principle!)

Ex: Let $b_t$ be a given (uncontrolled) sequence, $r$ a constant,

$$b_t > 0 \text{ all } t, \quad r \geq 0$$

and consider:

$$J_s(x_s) = \max_{C_s} \left\{ \sum_{t=s}^{T-1} b_t \sqrt{c_t} + b_T \sqrt{x_T} \right\}$$

where $x_{t+1} = r x_t - c_t$

→ For example, if $b_t = \beta^t$ ($\beta \in (0,1]$): interpretation as aggregated discounted utility from consuming $c_t$; horizon $T$ ⇝ consume everything at time $T$,

→ Note: $c_t \in [0, r x_t]$ otherwise ill-defined utility ($\sqrt{\text{neg.}}$)

Problem: Show that $J_s$ is of the form

$$J_s(x) = A_s \sqrt{x_s}$$

with $A_s > 0$ (not dep. on $x_s$).

Proof:

- True for $s = T$ (with $A_T = b_T$)
- Suppose true for $s = S$. Then for $S-1$:

$$J_{s-1}(x) = \max_{c \in [0, rx]} \left\{ b_{s-1} \sqrt{c} + A_s \sqrt{rx - c} \right\}$$

concave wrt $c$, maximized by

$$\frac{b_{s-1}}{2\sqrt{c}} = \frac{A_s}{2\sqrt{rx-c}}$$

$$\Updownarrow$$

$$\frac{rx - c}{c} = \left(\frac{A_s}{b_{s-1}}\right)^2$$

$$c = \underbrace{rx \cdot \left[ 1 + \left(\frac{A_s}{b_{s-1}}\right)^2 \right]^{-1}}_{\text{call this } \underbrace{Q_s}_{\substack{Q_s \text{ does not} \\ \text{depend on } x}} rx}$$

$$J_{s-1}(x) = b_{s-1} \sqrt{Q_s r} \sqrt{x} + A_s \sqrt{(1 - Q_s) r} \sqrt{x}$$

$$= \underbrace{\left( b_{s-1} \sqrt{Q_s} + A_s \sqrt{1 - Q_s} \right) \sqrt{r}}_{= A_{s-1}, \text{ indep of } x.} \sqrt{x}$$

Done!

The Euler equation (a.k.a Euler – Lagrange):

"Requires" $x_{t+1} = u_t$ and <u>internal</u> solution for this: is a 1st o.c.

$$\sum f(t, x_t, u_t) = \sum f(t, x_t, x_{t+1})$$

Notice: only two terms depend on $x_t$:

$$\ldots + f(t-1, x_{t-1}, x_t) + f(t, x_t, x_{t+1}) + \ldots$$

$$\frac{\partial}{\partial x_t} : \quad \boxed{f_3'(t-1, x_{t-1}, x_t) + f_2'(t, x_t, x_{t+1}) = 0}$$

Simplifies a lot if $f(t, x, u) = \beta^t F(x, u)$ (the <u>time-homogeneous</u> case);

then we obtain

$$F_u' + \beta F_x' = 0$$

The infinite-horizon time-homogeneous case; the Bellman equation.

Problem: $\sum_{t=s}^{\infty} \beta^t F(x_t, u_t)$

$$x_{t+1} = g(x_t, u_t)$$ ← no explicit "$t$"

Note: $$J_s(x) = \sum_{t=s}^{\infty} \beta^t F(x_t, u_t)$$

$$= \beta^s \sum_{\tilde{t}=0}^{\infty} \beta^{\tilde{t}} F(x_{\tilde{t}}, u_{\tilde{t}})$$

with new "clock" $\tilde{t}$,

$$= \beta^s \underbrace{J_0(x)}_{\text{call this "}J(x)\text{"}}.$$

D.P.: $$\beta^s J(x) = \max_u \left\{ \beta^s F(\ ) + \beta^{s+1} J(g(\ )) \right\}$$

yields

$$J(x) = \max_{u \in U} \left\{ F(x, u) + \beta J(g(x, u)) \right\}$$

Bellman cont'd:

Problem: no terminal time to start doing recursion.

Solution: Guess! Often it even works trying the finite-horizon case.

Ex: $\max_{q \in [0,r]} \sum \beta^t \sqrt{q x_t}$ s.t. $x_{t+1} = r x_t - q_t x_t$, $x_0 \geq 0$ given

... as previous one, just modified to rule out $x_t < 0$.

From the previous problem, it is tempting to <u>guess</u> $A\sqrt{x}$.

Try:

$$A\sqrt{x} = \max_{q \in [0,r]} \left\{ \sqrt{qx} + \beta A \sqrt{rx - qx} \right\}$$

$$= \sqrt{x} \max_{q \in [0,r]} \left\{ \sqrt{q} + \beta A \sqrt{r-q} \right\}$$

Works if $A = \max_{q \in [0,r]} \left\{ \sqrt{q} + \beta A \sqrt{r-q} \right\}$

then $$q = \frac{r}{1 + (A\beta)^2}$$

A bit of work. (Not very exam relevant.)