ECON4150 - Introductory Econometrics

Lecture 2: Review of Statistics

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Stock and Watson Chapter 2-3

Lecture outline

- Simple random sampling
- Distribution of the sample average
- Large sample approximation to the distribution of the sample mean
  - Law of large numbers
  - central limit theorem
- Estimation of the population mean
  - unbiasedness
  - consistency
  - efficiency
- Hypothesis test concerning the population mean
- Confidence intervals for the population mean

Simple random sampling

Simple random sampling means that n objects are drawn randomly from a population and each object is equally likely to be drawn

Let $Y_1, Y_2, ..., Y_n$ denote the 1st to the nth randomly drawn object.

Under simple random sampling:

- The marginal probability distribution of Y_i is the same for all $i = 1, 2, .., n$ and equals the population distribution of Y.
  - because $Y_1, Y_2, ..., Y_n$ are drawn randomly from the same population.
- Y_1 is distributed independently from $Y_2, ..., Y_n$
  - knowing the value of Y_i does not provide information on Y_j for $i \neq j$

When $Y_1, ..., Y_n$ are drawn from the same population and are independently distributed, they are said to be i.i.d random variables

Simple random sampling: Example

- Let G be the gender of an individual ($G = 1$ if female, $G = 0$ if male)
- G is a Bernoulli random variable with $E(G) = \mu_G = Pr(G = 1) = 0.5$
- Suppose we take the population register and randomly draw a sample of size n
  - The probability distribution of G_i is a Bernoulli distribution with mean 0.5
  - G_1 is distributed independently from $G_2, ..., G_n$
- Suppose we draw a random sample of individuals entering the building of the physics department
  - This is not a sample obtained by simple random sampling and $G_1, ..., G_n$ are not i.i.d
  - Men are more likely to enter the building of the physics department!

The sampling distribution of the sample average

The sample average $\bar{Y}$ of a randomly drawn sample is a random variable with a probability distribution called the sampling distribution.

$$\bar{Y} = \frac{1}{n}(Y_1 + Y_2 + ... + Y_n) = \frac{1}{n}\sum_{i=1}^{n} Y_i$$

Suppose $Y_1, ..., Y_n$ are i.i.d and the mean & variance of the population distribution of Y are respectively μ_Y & σ_Y^2

- The mean of $\overline{Y}$ is

$$E(\bar{Y}) = E\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i) = \frac{1}{n} nE(Y) = \mu_Y$$

- The variance of $\overline{Y}$ is

$$\begin{aligned} Var\left(\overline{Y}\right) &= Var\left(\tfrac{1}{n}\textstyle\sum_{i=1}^{n} Y_i\right) \\ &= \tfrac{1}{n^2}\textstyle\sum_{i=1}^{n} Var(Y_i) + 2\tfrac{1}{n^2}\textstyle\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} Cov(Y_i, Y_j) \\ &= \tfrac{1}{n^2} nVar(Y) + 0 \\ &= \tfrac{1}{n}\sigma_Y^2 \end{aligned}$$

The sampling distribution of the sample average:example

- Let G be the gender of an individual ($G = 1$ if female, $G = 0$ if male)
- The mean of the population distribution of G is

$$E\left(G\right) = \mu_G = p = 0.5$$

- The variance of the population distribution of G is

$$Var\left(G\right) = \sigma_G^2 = p(1-p) = 0.5(1-05) = 0.25$$

- The mean and variance of the average gender (proportion of women) $\overline{G}$ in a random sample with $n = 10$ are

$$E\left(\overline{G}\right) = \mu_G = 0.5$$

$$Var\left(\overline{G}\right) = \frac{1}{n}\sigma_G^2 = \frac{1}{10}0.25 = 0.025$$

The finite sample distribution of the sample average

The finite sample distribution is the sampling distribution that exactly describes the distribution of $\overline{Y}$ for any sample size n.

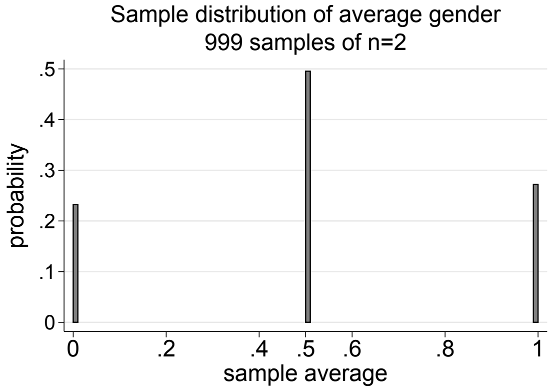
- In general the exact sampling distribution of $\overline{Y}$ is complicated and depends on the population distribution of Y.
- A special case is when $Y_1, Y_2, ..., Y_n$ are i.i.d draws from the $N\left(\mu_Y, \sigma_Y^2\right)$, because in this case

$$\overline{Y} \sim N\left(\mu_Y, \frac{\sigma_Y^2}{n}\right)$$

The finite sample distribution of average gender $\overline{G}$

Suppose we draw 999 samples of $n = 2$:

Sample 1			Sample 2			Sample 3			.....	Sample 999		
G_1	G_2	$\overline{G}$	G_1	G_2	$\overline{G}$	G_1	G_2	$\overline{G}$		G_1	G_2	$\overline{G}$
1	0	0.5	1	1	1	0	1	0.5		0	0	0



The asymptotic distribution of $\overline{Y}$

- Given that the exact sampling distribution of $\overline{Y}$ is complicated
- and given that we generally use large samples in econometrics
- we will often use an approximation of the sample distribution that relies on the sample being large

The asymptotic distribution is the approximate sampling distribution of $\overline{Y}$ if the sample size $n \longrightarrow \infty$

We will use two concepts to approximate the large-sample distribution of the sample average

- The law of large numbers.
- The central limit theorem.

Law of Large Numbers

The Law of Large Numbers states that if

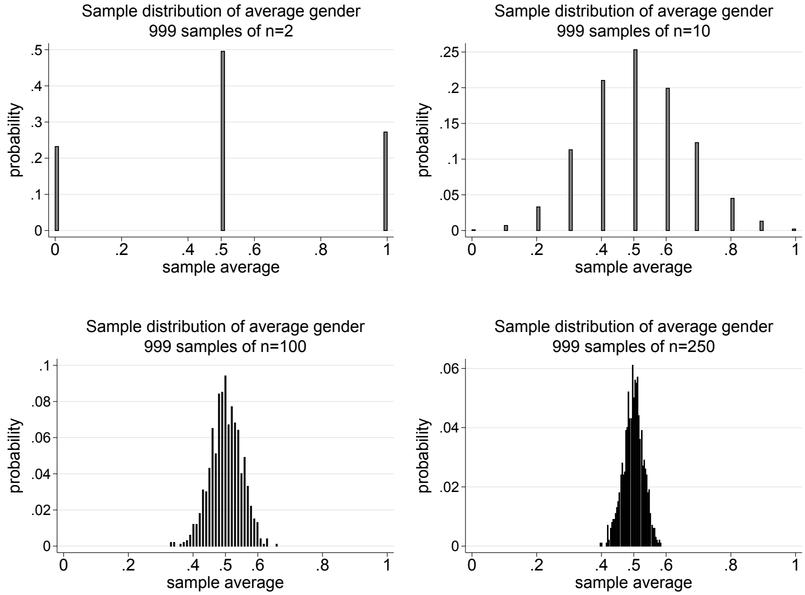
- Y_i, $i = 1, .., n$ are independently and identically distributed with $E(Y_i) = \mu_Y$
- and large outliers are unlikely; $Var(Y_i) = \sigma_Y^2 < \infty$

$\overline{Y}$ will be near μ_Y with very high probability when n is very large ($n \longrightarrow \infty$)

$$\overline{Y} \xrightarrow{p} \mu_Y$$

Law of Large Numbers

Example: Gender $G \sim \textit{Bernouilli}\,(0.5,\ 0.25)$



The Central Limit theorem

The Central Limit Theorem states that if

- Y_i, $i = 1, .., n$ are i.i.d. with $E(Y_i) = \mu_Y$
- and $Var(Y_i) = \sigma_Y^2$ with $0 < \sigma_Y^2 < \infty$

The distribution of the sample average is approximately normal if $n \longrightarrow \infty$

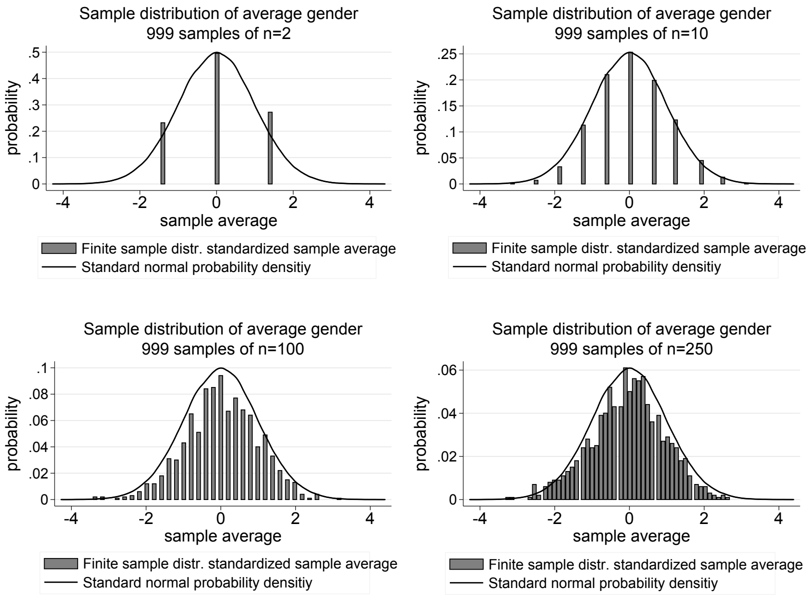
$$\overline{Y} \sim N\left(\mu_Y, \frac{\sigma_Y^2}{n}\right)$$

The distribution of the standardized sample average is approximately standard normal for $n \longrightarrow \infty$

$$\frac{\overline{Y} - \mu_Y}{\sigma_{\overline{Y}}^2} \sim N(0, 1)$$

The Central Limit theorem

Example: Gender $G \sim$ *Bernouilli* $(0.5,\ 0.25)$



The Central Limit theorem

How good is the large-sample approximation?

- If $Y_i \sim N\left(\mu_Y,\, \sigma_Y^2\right)$ the approximation is perfect
- If Y_i is not normally distributed the quality of the approximation depends on how close n is to infinity
- For $n \geq 100$ the normal approximation to the distribution of $\overline{Y}$ is typically very good for a wide variety of population distributions

Estimation

Estimators and estimates

An Estimator is a function of a sample of data *to be* drawn randomly from a population

- An estimator is a random variable because of randomness in drawing the sample

An Estimate is the numerical value of an estimator when it is actually computed using a specific sample.

Estimation of the population mean

Suppose we want to know the mean value of Y (μ_Y) in a population, for example

- The mean wage of college graduates.
- The mean level of education in Norway.
- The mean probability of passing the econometrics exam.

Suppose we draw a random sample of size n with $Y_1, ..., Y_n$ i.i.d

Possible estimators of μ_Y are:

- The sample average $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$
- The first observation Y_1
- The weighted average: $\widetilde{Y} = \frac{1}{n}\left(\frac{1}{2}Y_1 + \frac{3}{2}Y_2 + ... + \frac{1}{2}Y_{n-1} + \frac{3}{2}Y_n\right)$

Estimation of the population mean

To determine which of the estimators, $\overline{Y}$, Y_1 or $\widetilde{Y}$ is the best estimator of μ_Y we consider 3 properties:

Let $\hat{\mu}_Y$ be an estimator of the population mean μ_Y.

Unbiasedness: The mean of the sampling distribution of $\hat{\mu}_Y$ equals μ_Y

$$E\left(\hat{\mu}_Y\right) = \mu_Y$$

Consistency: The probability that $\hat{\mu}_Y$ is within a very small interval of μ_Y approaches 1 if $n \longrightarrow \infty$

$$\hat{\mu}_Y \xrightarrow{p} \mu_Y$$

Efficiency: If the variance of the sampling distribution of $\hat{\mu}_Y$ is smaller than that of some other estimator $\widetilde{\mu}_Y$, $\hat{\mu}_Y$ is more efficient

$$Var\left(\hat{\mu}_Y\right) < Var\left(\widetilde{\mu}_Y\right)$$

Example

Suppose we are interested in the mean wages μ_w of individuals with a master degree

We draw the following sample ($n = 10$) by simple random sampling

i	W_i
1	47281.92
2	70781.94
3	55174.46
4	49096.05
5	67424.82
6	39252.85
7	78815.33
8	46750.78
9	46587.89
10	25015.71

The 3 estimators give the following estimates:

$$\overline{W} = \frac{1}{10} \sum_{i=1}^{10} W_i = 52618.18$$

$$W_1 = 47281.92$$

$$\widetilde{W} = \frac{1}{10} \left(\frac{1}{2} W_1 + \frac{3}{2} W_2 + ... + \frac{1}{2} W_9 + \frac{3}{2} W_{10} \right) = 49398.82.$$

Unbiasedness

All 3 proposed estimators are unbiased:

- As shown on slide 5: $E\left(\overline{Y}\right) = \mu_Y$
- Since Y_i are i.i.d. $E\left(Y_1\right) = E\left(Y\right) = \mu_Y$
- $$\begin{aligned} E\left(\widetilde{Y}\right) &= E\left(\tfrac{1}{n}\left(\tfrac{1}{2}Y_1 + \tfrac{3}{2}Y_2 + ... + \tfrac{1}{2}Y_{n-1} + \tfrac{3}{2}Y_n\right)\right) \\ &= \tfrac{1}{n}\left(\tfrac{1}{2}E(Y_1) + \tfrac{3}{2}E(Y_2) + ... + \tfrac{1}{2}E(Y_{n-1}) + \tfrac{3}{2}E(Y_n)\right) \\ &= \tfrac{1}{n}\left[\left(\tfrac{n}{2}\cdot\tfrac{1}{2}\right)E\left(Y_i\right) + \left(\tfrac{n}{2}\cdot\tfrac{3}{2}\right)E\left(Y_i\right)\right] \\ & \quad E\left(Y_i\right) \qquad = \mu_Y \end{aligned}$$

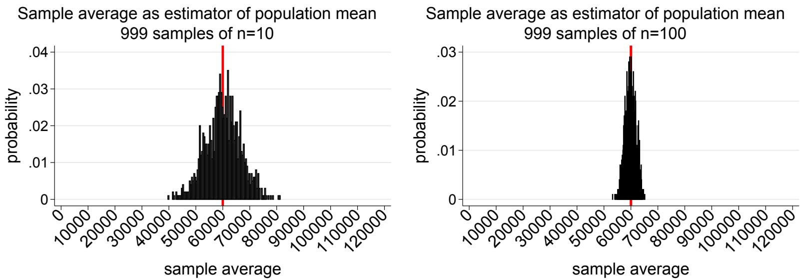
Consistency

Example: mean wages of individuals with a master degree with $\mu_W = 60\,000$

By the law of large numbers

$$\overline{W} \xrightarrow{p} \mu_W$$

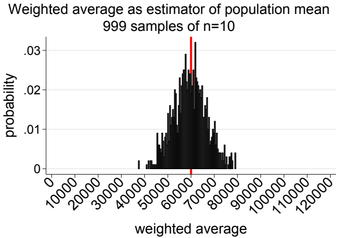
which implies that the probability that $\overline{W}$ is within a very small interval of μ_W approaches 1 if $n \longrightarrow \infty$

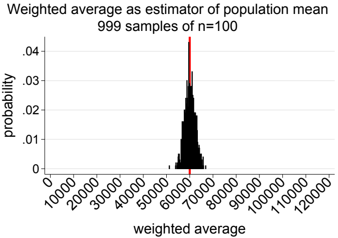


Consistency

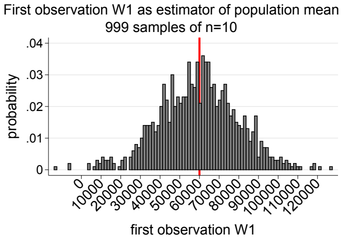
Example: mean wages of individuals with a master degree with $\mu_W = 60\,000$

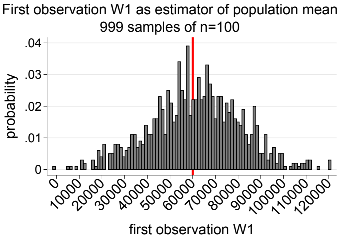
$\widetilde{W} = \frac{1}{n}\left(\frac{1}{2}W_1 + \frac{3}{2}W_2 + \dots + \frac{1}{2}W_{n-1} + \frac{3}{2}W_n\right)$ is also consistent





However W_1 is not a consistent estimator of μ_W:





Efficiency

Efficiency entails a comparison of estimators on the basis of their variance

- The variance of $\overline{Y}$ equals

$$Var\left(\overline{Y}\right) = \frac{1}{n}\sigma_Y^2$$

- The variance of Y_1 equals

$$Var\left(Y_1\right) = Var\left(Y\right) = \sigma_Y^2$$

- The variance of $\widetilde{Y}$ equals

$$Var\left(\widetilde{Y}\right) = 1.25\frac{1}{n}\sigma_Y^2$$

For any $n \geq 2$ $\overline{Y}$ is more efficient than Y_1 and $\widetilde{Y}$

BLUE: Best Linear Unbiased Estimator

- $\overline{Y}$ is not only more efficient than Y_1 and $\widetilde{Y}$, but it is more efficient than any unbiased estimator of μ_Y that is a weighted average of $Y_1, ..., Y_n$

$\overline{Y}$ is the Best Linear Unbiased Estimator (BLUE) it is the most efficient estimator of μ_Y among all unbiased estimators that are weighted averages of $Y_1, ..., Y_n$

- Let $\hat{\mu}_Y$ be an unbiased estimator of μ_Y

$$\hat{\mu}_Y = \frac{1}{n}\sum_{i=1}^{n} a_i Y_i \qquad \textit{with } a_1, ...a_n \textit{ nonrandom constants}$$

then $\overline{Y}$ is more efficient than $\hat{\mu}_Y$, that is

$$Var\left(\overline{Y}\right) < Var\left(\hat{\mu}_Y\right)$$

Hypothesis tests concerning the population mean

Hypothesis tests concerning the population mean

Consider the following questions:

- Is the mean monthly wage of college graduates equal to NOK 60 000?
- Is the mean level of education in Norway equal to 12 years?
- Is the mean probability of passing the econometrics exam equal to 1?

These questions involve the population mean taking on a specific value $\mu_{Y,0}$

Answering these questions implies using data to compare a null hypothesis

$$H_0 : \; E(Y) = \mu_{Y,0}$$

to an alternative hypothesis, which is often the following two sided hypothesis

$$H_1 : \; E(Y) \neq \mu_{Y,0}$$

Hypothesis tests concerning the population mean
p-value

Suppose we have a sample of n i.i.d observations and compute the sample average $\overline{Y}$

The sample average can differ from $\mu_{Y,0}$ for two reasons

1. The population mean μ_Y is not equal to $\mu_{Y,0}$ (H_0 not true)
2. Due to random sampling $\overline{Y} \neq \mu_Y = \mu_{Y,0}$ (H_0 true)

To quantify the second reason we define the p-value

The p-value is the probability of drawing a sample with $\overline{Y}$ at least as far from $\mu_{Y,0}$ given that the null hypothesis is true.

Hypothesis tests concerning the population mean
p-value

$$p - value = Pr_{H_0}\left[\,|\overline{Y} - \mu_{Y,0}| \;>\; |\overline{Y}^{act} - \mu_{Y,0}|\,\right]$$

To compute the p-value we need to know the sampling distribution of $\overline{Y}$

- Sampling distribution of $\overline{Y}$ is complicated for small n
- With large n the central limit theorem states that

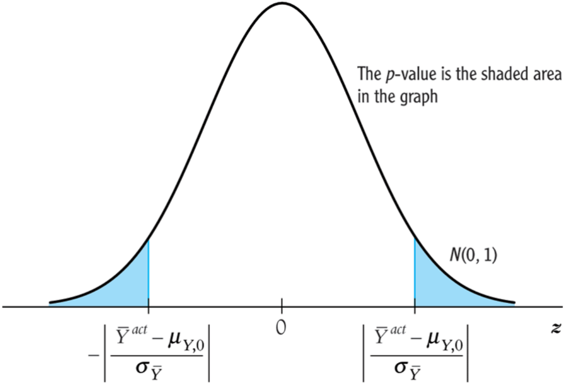
$$\overline{Y} \sim N\left(\mu_Y,\ \frac{\sigma_Y^2}{n}\right)$$

- This implies that if the null hypothesis is true:

$$\frac{\overline{Y} - \mu_{Y,0}}{\sqrt{\frac{\sigma_Y^2}{n}}} \sim N(0,1)$$

Computing the p-value when σ_Y is known

$$p - value = Pr_{H_0}\left[\left|\frac{\overline{Y}-\mu_{Y,0}}{\sqrt{\frac{\sigma_Y^2}{n}}}\right| > \left|\frac{\overline{Y}^{act}-\mu_{Y,0}}{\sqrt{\frac{\sigma_Y^2}{n}}}\right|\right] = 2\Phi\left(-\left|\frac{\overline{Y}^{act}-\mu_{Y,0}}{\sqrt{\frac{\sigma_Y^2}{n}}}\right|\right)$$



- For large n, p-value = the probability that Z falls outside $\left|\frac{\overline{Y}^{act}-\mu_{Y,0}}{\sqrt{\frac{\sigma_Y^2}{n}}}\right|$

Estimating the standard deviation of $\overline{Y}$

- In practice σ_Y^2 is usually unknown and must be estimated

The sample variance s_Y^2 is the estimator of $\sigma_Y^2 = E\left[(Y_i - \mu_Y)^2\right]$

$$s_Y^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(Y_i - \overline{Y}\right)^2$$

- division by $n-1$ because we "replace" μ_Y by $\overline{Y}$ which uses up 1 degree of freedom
- if $Y_1, ..., Y_n$ are i.i.d. and $E\left(Y^4\right) < \infty$, $s_Y^2 \xrightarrow{p} \sigma_Y^2$ (Law of Large Numbers)

The sample standard deviation $s_Y = \sqrt{s_Y^2}$ is the estimator of σ_Y

Computing the p-value using $SE\left(\overline{Y}\right) = \widehat{\sigma}_{\overline{Y}}$

The standard error $SE(\overline{Y})$ is an estimator of $\sigma_{\overline{Y}}$

$$SE\left(\overline{Y}\right) = \frac{s_Y}{\sqrt{n}}$$

- Because s_Y^2 is a consistent estimator of σ_Y^2, we can (for large n) replace $\sqrt{\frac{\sigma_Y^2}{n}}$ by $SE\left(\overline{Y}\right) = \frac{s_Y}{\sqrt{n}}$
- This implies that when σ_Y^2 is unknown and $Y_1, ..., Y_n$ are i.i.d. the p-value is computed as

$$p - value = 2\Phi\left(-\left|\frac{\overline{Y}^{act} - \mu_{Y,0}}{SE\left(\overline{Y}\right)}\right|\right)$$

The t-statistic and its large-sample distribution

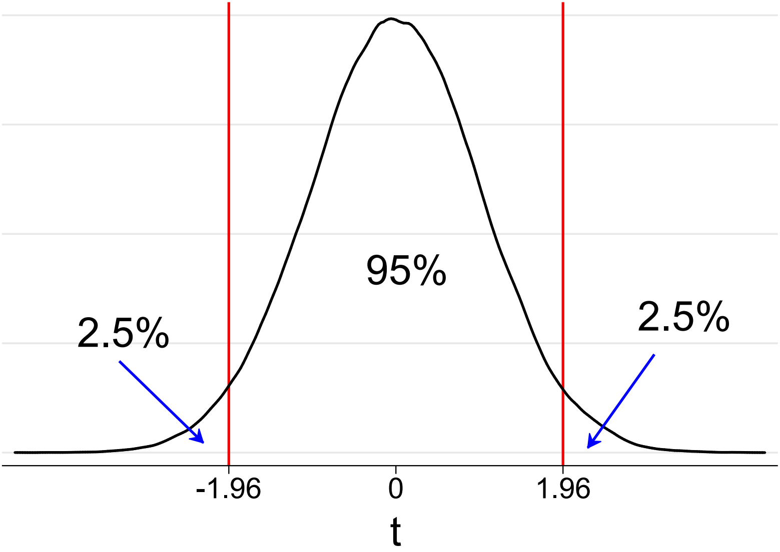
- The standardized sample average $\left(\overline{Y}^{act} - \mu_{Y,0}\right) / SE\left(\overline{Y}\right)$ plays a central role in testing statistical hypothesis
- It has a special name, the **t-statistic**

$$t = \left| \frac{\overline{Y} - \mu_{Y,0}}{SE\left(\overline{Y}\right)} \right|$$

- t is approximately $N(0,1)$ distributed for large n
- The p-value can be computed as

$$p - value = 2\Phi\left(-\left|t^{act}\right|\right)$$

The t-statistic and its large-sample distribution



Type I and type II errors and the significance level

There are 2 types of mistakes when conduction a hypothesis test

Type I error refers to the mistake of rejecting H_0 when it is true

Type II error refers to the mistake of not rejecting H_0 when it is false

- In hypothesis testing we usually fix the probability of a type I error

The significance level α is the probability of rejecting H_0 when it is true

- Most often used significance level is 5% ($\alpha = 0.05$)

Since area in tails of $N(0, 1)$ outside ± 1.96 is 5%:

- We reject H_0 if p-value is smaller than 0.05.
- We reject H_0 if $\left|t^{act}\right| > 1.96$

4 steps in testing a hypothesis about the population mean

$$H_0 : E(Y) = \mu_{Y,0} \qquad H_1 : E(Y) \neq \mu_{Y,0}$$

Step 1: Compute the sample average $\overline{Y}$

Step 2: Compute the standard error of $\overline{Y}$

$$SE\left(\overline{Y}\right) = \frac{s_Y}{\sqrt{n}}$$

Step 3: Compute the t-statistic

$$t^{act} = \frac{\overline{Y} - \mu_{Y,0}}{SE\left(\overline{Y}\right)}$$

Step 4: Reject the null hypothesis at a 5% significance level if

- $|t^{act}| > 1.96$
- or if $p - value < 0.05$

Hypothesis tests concerning the population mean

Example: The mean wage of individuals with a master degree

Suppose we would like to test

$$H_0 : \; E(W) = 60000 \qquad H_1 : \; E(W) \neq 60000$$

using a sample of 250 individuals with a master degree

Step 1: $\overline{W} = \frac{1}{n} \sum_{i=1}^{n} W_i = 61977.12$

Step 2: $SE\left(\overline{W}\right) = \frac{s_W}{\sqrt{n}} = 1334.19$

Step 3: $t^{act} = \frac{\overline{W} - \mu_{W,0}}{SE\left(\overline{W}\right)} = \frac{61977.12 - 60000}{1334.19} = 1.48$

Step 4: Since we use a 5% significance level, we do not reject H_0 because $|t^{act}| = 1.48 < 1.96$

Note: We do never *accept* the null hypothesis!

Hypothesis tests concerning the population mean

Example: The mean wage of individuals with a master degree

This is how to do the test in Stata:

```
                                      ___  ____  ____  ____  ____(R)
                                     /__    /   ____/   /   ____/
                                    ___/   /   /___/   /   /___/
                                      Statistics/Data Analysis

. ttest wage=60000

One-sample t test
------------------------------------------------------------------------------
Variable |     Obs        Mean    Std. Err.   Std. Dev.   [95% Conf. Interval]
---------+--------------------------------------------------------------------
    wage |     250    61977.12    1334.189    21095.37    59349.39    64604.85
------------------------------------------------------------------------------
    mean = mean( wage)                                            t =   1.4819
Ho: mean = 60000                                 degrees of freedom =      249

   Ha: mean < 60000            Ha: mean != 60000             Ha: mean > 60000
 Pr(T < t) = 0.9302         Pr(|T| > |t|) = 0.1396          Pr(T > t) = 0.0698
```

Hypothesis test with a one-sides alternative

- Sometimes the alternative hypothesis is that the mean exceeds $\mu_{Y,0}$

$$H_0 : E(Y) = \mu_{Y,0} \qquad H_1 : E(Y) > \mu_{Y,0}$$

- In this case the p-value is the area under $N(0,1)$ to the right of the t-statistic

$$p - value = Pr_{H_0}\left(t > t^{act}\right) = 1 - \Phi\left(t^{act}\right)$$

- With a significance level of 5% ($\alpha = 0.05$) we reject H_0 if $t^{act} > 1.64$

- If the alternative hypothesis is $H_1 : E(Y) < \mu_{Y,0}$

$$p - value = Pr_{H_0}\left(t < t^{act}\right) = 1 - \Phi\left(-t^{act}\right)$$

and we reject H_0 if $t^{act} < -1.64$ / $p - value < 0.05$

Hypothesis test with a one-sides alternative
Example: The mean wage of individuals with a master degree

```
  ___  ____  ____  ____  ____(R)
 /__    /   ____/   /   ____/
___/   /   /___/   /   /___/
  Statistics/Data Analysis

. ttest wage=60000

One-sample t test
------------------------------------------------------------------------------
Variable |     Obs        Mean    Std. Err.   Std. Dev.   [95% Conf. Interval]
---------+--------------------------------------------------------------------
    wage |     250    61977.12    1334.189    21095.37    59349.39    64604.85
------------------------------------------------------------------------------
    mean = mean( wage)                                            t =   1.4819
Ho: mean = 60000                                 degrees of freedom =      249

    Ha: mean < 60000             Ha: mean != 60000             Ha: mean > 60000
 Pr(T < t) = 0.9302         Pr(|T| > |t|) = 0.1396          Pr(T > t) = 0.0698
```

Confidence intervals for the population mean

- Suppose we would do a two-sided hypothesis test for many different values of $\mu_{Y,0}$
- On the basis of this we can construct a set of values which are not rejected at a 5% significance level
- If we were able to test all possible values of $\mu_{Y,0}$ we could construct a 95% confidence interval

A 95% confidence interval is an interval that contains the true value of μ_Y in 95% of all possible random samples.

- Instead of doing infinitely many hypothesis tests we can compute the 95% confidence interval as

$$\left\{ \overline{Y} - 1.96 \cdot SE\left(\overline{Y}\right) \quad , \quad \overline{Y} + 1.96 \cdot SE\left(\overline{Y}\right) \right\}$$

- Intuition: a value of $\mu_{Y,0}$ smaller than $\overline{Y} - 1.96 \cdot SE\left(\overline{Y}\right)$ or bigger than $\overline{Y} - 1.96 \cdot SE\left(\overline{Y}\right)$ will be rejected at $\alpha = 0.05$

Confidence intervals for the population mean

Example: The mean wage of individuals with a master degree

When the sample size *n* is large:

$$95\% \text{ confidence interval for } \mu_Y = \left\{ \overline{Y} \pm 1.96 \cdot SE\left(\overline{Y}\right) \right\}$$

$$90\% \text{ confidence interval for } \mu_Y = \left\{ \overline{Y} \pm 1.64 \cdot SE\left(\overline{Y}\right) \right\}$$

$$99\% \text{ confidence interval for } \mu_Y = \left\{ \overline{Y} \pm 2.58 \cdot SE\left(\overline{Y}\right) \right\}$$

Using the sample of 250 individuals with a master degree:

95% conf. int. for μ_W is

$$\{61977.12 \pm 1.96 \cdot 1334.19\} = \{59349.39\ ,\ 64604.85\}$$

90% conf. int. for μ_W is

$$\{61977.12 \pm 1.64 \cdot 1334.19\} = \{59774.38\ ,\ 64179.86\}$$

99% conf. int. for μ_W is

$$\{61977.12 \pm 2.58 \cdot 1334.19\} = \{58513.94\ ,\ 65440.30\}$$

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