Deb and Frank have the following utility functions:
$U_D = x_D + 2y_D$ and $U_F = x_F^{\alpha} y_F^{1-\alpha}$ with $\alpha \in (0,1)$.
Their endowment is one unit of good y for Deb and one unit of good x for Frank.

**a) [8%] Determine the expenditure function for Frank. What happens with the expenditure if all prices are multiplied by the same number?**

Minimize the expenditure s.t. a utility constrain. The Lagrangean is:

$$\mathcal{L} = p_x x_F + p_y y_F - \lambda \left( x_F^{\alpha} y_F^{1-\alpha} - \bar{u}_F \right)$$

The solution is interior. The FOCs are (I drop the asterisc for notational convenience):

$$\begin{cases} p_x = \lambda \alpha \frac{1}{x_F} x_F^{\alpha} y_F^{1-\alpha} \\ p_Y = \lambda (1-\alpha) \frac{1}{y_F} x_F^{\alpha} y_F^{1-\alpha} \\ x_F^{\alpha} y_F^{1-\alpha} = \bar{u}_F \end{cases}$$

Taking the ratio of the first two and rearranging gives:

$$\frac{1-\alpha}{\alpha} \frac{p_x}{p_y} x_F = y_F$$

Substitute in the constraint to get:

$$x_F^{\alpha} \left( \frac{1-\alpha}{\alpha} \frac{p_x}{p_y} x_F \right)^{1-\alpha} = \bar{u}_F$$

which gives the Hicksian demand of good $x$:

$$x_F = \bar{u}_F \left( \frac{\alpha}{1-\alpha} \frac{p_y}{p_x} \right)^{1-\alpha}$$

The Hicksian demand of good $y$ is:

$$y_F = \frac{1-\alpha}{\alpha} \frac{p_x}{p_y} \bar{u}_F \left( \frac{\alpha}{1-\alpha} \frac{p_y}{p_x} \right)^{1-\alpha} = \bar{u}_F \left( \frac{\alpha}{1-\alpha} \frac{p_y}{p_x} \right)^{-\alpha}$$

The expenditure function is:

$$E_F = p_x \bar{u}_F \left( \frac{\alpha}{1-\alpha} \frac{p_y}{p_x} \right)^{1-\alpha} + p_y \bar{u}_F \left( \frac{\alpha}{1-\alpha} \frac{p_y}{p_x} \right)^{-\alpha}$$

$$E_F = \bar{u}_F p_x^{\alpha} p_y^{1-\alpha} \left[ \left( \frac{\alpha}{1-\alpha} \right)^{1-\alpha} + \left( \frac{\alpha}{1-\alpha} \right)^{-\alpha} \right]$$

Clearly, when multiplying the prices by a constant $k > 0$, the expenditure increases by the same constant $k$.

**b) [5%] What is the elasticity of substitution for Deb? what is the implication for the income and substitution effects?**

The elasticity of substitution for Deb is infinite.

The price effect is equivalent to the income effect. The substitution effect is 0.

**c) [5%] Suppose that x and y measure the same commodity in two different states of the world, say states X and Y. Rewrite the utility functions as expected utility functions. For which $\alpha$ do the utility functions for D and F have the same MRS at the 45 degree line? What does this mean?**

The expected utility functions for the agents are:

$$U_D = \frac{1}{3}x_D + \frac{2}{3}y_D$$

and

$$U_F = \alpha \ln x_F + (1 - \alpha) \ln y_F$$

with $\alpha \in (0, 1)$.

Clearly, when $\alpha = \frac{1}{3}$, the individuals' MRS at the 45 degree line is the same.

This means that individuals have the same belief about the likelihood of the two states $X$ and $Y$.

**d) [5%] Compute Deb's certainty equivalent at her endowment. Compute the Arrow-Pratt index of absolute risk aversion for Frank.**

The Arrow-Pratt index of absolute risk aversion is:

$$R_a(w) \equiv -\frac{u''(w)}{u'(w)}.$$

The cardinal utility of Frank is $ln$. Thus:

$$R_a(w) \equiv -\frac{-\frac{1}{w^2}}{\frac{1}{w}} = \frac{1}{w}.$$

**e) [10%] Compute the Walrasian equilibrium for this economy.**

Since Deb has perfectly substitutable preferences, a corner solution would emerge unless relative prices are $\frac{p_x}{p_y} = \frac{1}{2}$. But a corner solution is never optimal for Frank. Thus, equilibrium prices must satisfy this condition. Let $p_x = 1$ and $p_y = 2$.

Next, the cost of the equilibrium bundle for Deb has to pass through her endowment vector.

$$p_x x_D + p_y y_D = p_x 0 + p_y 1 = 2$$

Thus,

$$x_D + 2y_D = 1$$

By feasibility, the budget constraint for Frank is:

$$(1 - x_F) + 2(1 - y_F) = 1$$

Thus, maximize the utility of Frank subject tot his budget constraint:

$$\mathcal{L} = x_F^{\alpha} y_F^{1-\alpha} + \mu\left((1 - x_F) + 2(1 - y_F) - 1\right)$$

The solution is interior. The FOCs are (I drop the asterisc for notational convenience):

$$\begin{cases} \mu = \alpha \frac{1}{x_F} x_F^{\alpha} y_F^{1-\alpha} \\ 2\mu = (1-\alpha) \frac{1}{y_F} x_F^{\alpha} y_F^{1-\alpha} \\ (1 - x_F) + 2(1 - y_F) = 2 \end{cases}$$

Taking the ratio of the first two and rearranging gives:

$$\frac{1-\alpha}{\alpha} \frac{1}{2} x_F = y_F$$

Substitute in the constraint to get:

$$x_F = \alpha$$

$$y_F = \frac{1-\alpha}{2}.$$

Thus,

$$x_D = 1 - \alpha$$

$$u_D = \frac{1+\alpha}{2}.$$

1b) To produce electricity E, firm HydroP uses water W and a plant P as main inputs. It operates in a unique location, so that no further plants can be built. Without the plant the production is 0. With the plant, electricity can be produced according to the following production function:

$$E = \begin{cases} 0 & \text{if } W \leq \underline{W} \\ 4W & \text{if } \underline{W} \leq W \leq \bar{W} \\ 3\bar{W} & \text{if } \bar{W} < W \end{cases}$$

**a) [5%] Is this production function: continuous? (strictly) increasing? (strictly) quasiconcave? increasing/decreasing/constant returns to scale?**

The production function is not continuous as it exhibits a jump at $W = \bar{W}$ and at $W = \underline{W}$.

Since the jump is downwards, it is not increasing (nor strictly increasing).

It is quasi-concave, but not strictly (it is constant for $W > \bar{W}$).

The function is first constant returns to scale $W < \underline{W}$, then increasing at the first jump, then again CRS, then decreasing from the second jump onwards. The answer none of those is sufficient.

**b) [7%] Determine the cost function for this firm (let the price of water be and let K be the cost of the plant). Is the cost function continuous?**

To find the cost function, fix a level of production $E$ and the prices $p_w$ and $K$.

The cost of producing $E$ is given by:

$$C(E, p_w, K) = \begin{cases} 0 & \text{if } E = 0 \\ K + p_w \underline{W} & \text{if } 0 < E \leq 4\underline{W} \\ K + p_w \frac{1}{4} E & \text{if } 4\underline{W} < E \leq 4\bar{W} \end{cases}$$

note that it is impossible to produce more than $W = 4\bar{W}$. The function is not continuous as it has a jump at 0 (due to the fixed cost).

**c) [5%] Can one recover the original production function from the cost function? Why not?**

It is not possible to recover the original production function from the cost function. It is sufficient to see that inputs $W > \bar{W}$ are never cost minimizing, so this part of the production function cannot be recovered. The necessary conditions are:

**Exam in ECON3220/4220 Fall 2019**

**Problem 2 (in game theory and the economics of information; weight 50%)**

*Guide for grading*

Consider a strategic situation between an *insurance company* (*U*) and an *individual* (*I*). *U* can either offer *cheap* (*C*) or *expensive* (*E*) insurance. *I* can either accept a *deductible* (*D*) or insist on *no deductible* (*N*). *I* can be of two types: either he is inherently *careful* (*F*) or he is inherently *careless* (*L*); this is not a choice for *I*. The players' payoffs depending on their actions and *I*'s type is shown below. (Payoffs are zero for both players if *U* offers *E* to *F* since *F* does not accept expensive insurance.)

*F*

| | *D* | *N* |
|---|---|---|
| *C* | 2, 2 | 2, 4 |
| *E* | 0, 0 | 0, 0 |

*L*

| | *D'* | *N'* |
|---|---|---|
| *C* | –2, 0 | –4, 4 |
| *E* | 2, –2 | 2, 2 |

(a) For each of these games, determine the set of rationalizable strategies for each player, and find the Nash equilibrium/a.

*In the left game: C is rationalizable for U, N is rationalizable for I, and (C, N) is the unique Nash equilibrium. In the right game: E is rationalizable for U, N' is rationalizable for I, and (E, N') is the unique Nash equilibrium.*

(b) Assume next that only *I* knows his own type (so that *I* is informed), while player *U* thinks that the two types of *I* are equally likely (so that *U* is uninformed). Model this situation in an ex ante perspective by specifying the Bayesian normal form.

| | *DD'* | *DN'* | *ND'* | *NN'* |
|---|---|---|---|---|
| *C* | 0, 1 | –1, 3 | 0, 2 | –1, 4 |
| *E* | 1, –1 | 1, 1 | 1, –1 | 1, 1 |

(c) For the Bayesian normal form found in part (b), determine the set of rationalizable strategies for each player, and find the Nash equilibrium/a.

*E is rationalizable for U, DN' and NN' are rationalizable for I, and (E, DN') and (E, NN') are the two pure-strategy Nash equilibrium. There is also a continuum of mixed strategy Nash equilibria where U chooses E and I mixes between DN' and NN'.*

(d) Assume now that *I* acts before *U*, and that *I*'s choice of *D* or *N* can be observed by *U* before she makes her choice of *C* or *E*. Show that there are two kinds of perfect Bayesian equilibria:

one separating equilibrium and one pooling equilibrium. Which of these are more reasonable?

*The candidate is expected to draw the extensive form. Note that I of type L always prefers to choose N′. Therefore, the probability of F if N/N′ is chosen cannot exceed ½ and U chooses E after N/N′. The separating equilibrium has I of type F choosing D and U choosing C after D/D′. This is sequentially optimal for both given believes that satisfy Bayes' rule. The pooling equilibrium has I of type F choosing N and U choosing E after D/D′. This is sequentially rationally for both given that U assigns no more than 2/3 probability on F if D/D′ is chosen. Although Bayes' rule cannot be applied, and thus such a belief is permissible, it is not reasonable since I of type L always prefers to choose N′. Therefore, the separating equilibrium is more reasonable.*

(e) Assume now that *U* acts before *I*, where *U* can choose between the four strategies *CC′*, *CE′*, *EC′* and *EE′*. The first letter corresponds to what *U* will do if *I* chooses *D*, and the second letter corresponds to what *U* will do if *I* chooses *N*. (So *EC′* is a commitment that the insurance company will offer expensive insurance if the individual accepts a deductible and will offer cheap insurance if the individual insists on no deductible.) The choice of *CC′*, *CE′*, *EC′* or *EE′* can be observed by *I* before he makes his own choice. Show that there is a unique subgame perfect Nash equilibrium outcome of this game.

*The candidate is expected to draw the extensive form. It follows from backward induction that U chooses CE′, that I of type F chooses D if CE′ is observed and that I of type L chooses N if CE′ is observed. This determines the unique subgame perfect Nash equilibrium outcome. A good candidate will observe that there is not a unique subgame perfect Nash equilibrium, as I of type F is indifferent between D and N if EE′ is observed.*