Solution sketches for
the exam in ECON3220/4220, Fall 2021

Problem 2 (Game theory and the economics of information)

Part A: Nash and Bayesian Nash equilibrium

Weight: 30% *(with equal weight* $= 3\%$ *on each subproblem)*

Consider a simultaneous move duopoly game with homogeneous goods in which two firms 1 and 2 compete by selecting quantities q_1 and q_2 respectively and where the price is determined by the inverse demand function given by: $p = 60 - q_1 - q_2$. Firm 1's cost is 18 per unit of production. Firm 2's cost is c_2 per unit of production. There are no fixed costs and each firm maximizes its own profits.

(a) Write down the profit function of each firm.

$\pi_1(q_1, q_2) = (60 - q_1 - q_2 - 18)q_1$ *and* $\pi_2(q_1, q_2) = (60 - q_1 - q_2 - c_2)q_2$.

(b) Without reference to this specific problem, explain what a best response function is, what a Nash Equilibrium is, and what a rationalizable strategy is.

See Watson's Strategy or the lecture notes.

(c) The best response function for each of the firms is given by

$$q_1 = \frac{60 - q_2 - 18}{2} \qquad \text{and} \qquad q_2 = \frac{60 - q_1 - c_2}{2}.$$

Show how each of these functions are derived from the corresponding profit function.

The first order conditions are $60 - q_1 - q_2 - 18 - q_1 = 0$ *and* $60 - q_1 - q_2 - c_2 - q_2 = 0$. *Solving wrt.* q_1 *and* q_2 *yields the best response functions.*

Assume for subproblems (d), (e), and (f) that $c_2 = 18$ and that this is commonly known among the firms.

(d) What quantities for each of the firms can be best responses to some belief about the quantity of the other firm.

The quantity of each firm i, q_i, *must be the interval* $[0, 21]$ *in order to be a non-negative quantity that is best response to a belief about a non-negative quantity for the opponent.*

(e) Solve for the Nash Equilibrium.

The quantities $q_1 = q_2 = 14$ *is the unique solutions to the set of equations given by the best response functions.*

(f) What quantities for each of the firms are rationalizable?

If a firm believes that the other firm chooses a best response to some belief about its own quantity, then it believes that the other firm chooses some quantity in the interval $[0, 21]$. *Quantities in the interval* $[10.5, 21]$ *can be a best response to such a belief. Continuing the iteration leads to the conclusion that only the Nash equilibrium quantity is rationalizable. Further calculations are not expected.*

Assume for subproblems (g), (h), (i), and (j) that firm 1 is unsure about c_2 but believes with probability $\frac{1}{2}$ that $c_2 = 12$ and with probability $\frac{1}{2}$ that $c_2 = 24$. Assume furthermore that firm 2 knows whether $c_2 = 12$ or $c_2 = 24$ and also knows firm 1's beliefs.

(g) Specify the profit function for firm 1, and calculate its best response function under the assumption that firm 1 is risk-neutral and thus maximizes expected profit.

Let q_2^{12} *be the quantity chosen by firm* 2 *with cost* 12 *and* q_2^{24} *the quantity chosen by firm* 2 *with cost* 24. *The profit function is* $\pi_1(q_1, q_2^{12}, q_2^{24}) = \frac{1}{2}(60 - q_1 - q_2^{12} - 18)q_1 + \frac{1}{2}(60 - q_1 - q_2^{24} - 18)q_1 = \big(60 - q_1 - \frac{1}{2}(q_2^{12} + q_2^{24}) - 18\big)q_1$, *leading to the following best response function:*

$$q_1 = \frac{60 - \frac{1}{2}(q_2^{12} + q_2^{24}) - 18}{2} .$$

(h) Specify the best response function for each type of firm 2.

$$q_2^{12} = \frac{60 - q_1 - 12}{2} \qquad \textit{and} \qquad q_2^{24} = \frac{60 - q_1 - 24}{2} .$$

(i) Explain in words what a Bayesian Nash Equilibrium is in this game.

See Watson's Strategy or the lecture notes.

(j) Solve for the Bayesian Nash Equilibrium.

The quantities $q_1 = 14$, $q_2^{12} = 17$, *and* $q_2^{24} = 11$ *is the unique solutions to the set of equations given by the best response functions.*

Part B: Adverse section

Weight: 20% *(with equal weight* = 5% *on each subproblem)*

(a) Consider the model of an insurance market presented in Section 8.1.3 in Jehle and Reny (2011). Describe the pure strategy subgame-perfect equilibrium in this model, if such an equilibrium exist.

The insurance companies offer a menu of contracts consisting of one contract intended for high-risk consumers and one contract intended for low-risk consumers. The high-risk contract is accepted by the high-risk consumers and yields zero profit for the companies when sold to them, and the low-risk contract is accepted by the low-risk consumers and yields zero profit for the companies when sold to them. The high-risk contract provides full insurance, while the low-risk contract provides only partial insurance in order to avoid that high-risk consumers choose the low-risk contract. The high-risk consumers are indifferent between the two types of contracts.

(b) In this model, why is the equilibrium necessarily separating, if it exists? Why does not such an equilibrium exist if the fraction α of low-risk consumers is sufficiently large?

A pooling equilibrium cannot exist because it can be undermined by an insurance company profitable cream-skimming contract that provides less insurance and with a premium making it attractive only for the low-risk consumers. If the fraction α of low-risk consumers is sufficiently large, an insurance company can offer a full insurance contract which is better for both types of consumers and yields the company positive profit. The undermines the candidate for subgame-perfect equilibrium described under part (a).

(c) The model of Section 8.1.3 in Jehle and Reny (2011) is presented in the context of motor insurance, but it provides insights also for health insurance. Assume that health insurance companies are only allowed to offer contracts that fully insure the consumers against the cost of treatment. Consumers' probability of falling ill is private information, and it is either low or high. What two equilibrium outcomes can arise?

One equilibrium outcome where both types of consumers are fully insured arises if the premium needed for zero profit of providing full insurance when both type of consumers accept the contract makes the contract preferable for low-risk consumer as compared to not buying insurance. Otherwise, the equilibrium outcome is characterized by a full insurance contract accepted only by the high-risk consumers at a premium needed to yield zero profit when sold only to them, while the low-risk consumers do not buy insurance.

(d) The Affordable Care Act (Obamacare) in the United States (i) requires that essential health benefits must be provided and (ii) imposes an individual mandate requiring everyone to have insurance or pay a penalty. In context of the model described in Section 8.1.3 in Jehle and Reny (2011), explain the rationale behind these two provisions of the Affordable Care Act.

Provision (i) outlaws offers of partial insurance that might lead to the outcome described in part (a). Provision (ii) makes it worse for low-risk consumers to remain uninsured, thereby improving the possibility of reaching the first of the two equilibrium outcomes described in part (c), yielding full insurance for all.

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Problem 1 (Microeconomic theory). Weights are as specified.

Part A (20%):

Andrea has the following indirect utility function:

$$V\left(p_1, p_2, m\right)=\frac{m}{p_1+p_2}.$$

A1 (4%): Derive Andrea's Marshallian demands of the goods x_1 and x_2.

SA1: Using Roy's identity,

$$x_1\left(p_1, p_2, m\right)=-\frac{\frac{\partial V\left(p_1, p_2, m\right)}{\partial p_1}}{\frac{\partial V\left(p_1, p_2, m\right)}{\partial m}}=-\frac{-m\left(p_1+p_2\right)^{-2}}{\left(p_1+p_2\right)^{-1}}=\frac{m}{p_1+p_2}$$

The same demand holds also for x_2.

A2 (4%): Briefly discuss the complementarity/substitutability of the goods.

SA2: Andrea purchases equal quantities of the two goods for any combination of prices. Thus, the goods are perfect complements.

A3 (4%): Use the relationship between indirect utility and expenditure function to find the expenditure function and, then, the Hicksian demands.

SA3: The relationship is:

$$V\left(p_1, p_2, e\left(p_1, p_2, u\right)\right)=u.$$

Substituting gives:

$$\frac{e\left(p_1, p_2, u\right)}{p_1+p_2}=u$$

and, rearranging,

$$e\left(p_1, p_2, u\right)=u\left(p_1+p_2\right).$$

The Hicksian demands are the derivative of $e\left(p_1, p_2, u\right)$ with respect to prices:

$$x_1^h\left(p_1, p_2, u\right)=\frac{\partial e\left(p_1, p_2, u\right)}{\partial p_1}=\frac{\partial\left[u\left(p_1+p_2\right)\right]}{\partial p_1}=u.$$

Again, the same demand holds also for x_2^h.

A4 (4%): Draw the indifference curves of Andrea and discuss its properties.

SA4: These are Leontief preferences (L-shaped) with corner on the 45 degree line.

These satisfy completeness, transitivity, continuity, strict monotonicity, and convexity (but not strict convexity).

A5 (4%): Discuss the effect of taxation of good 1 for Andrea.

SA5: First, the elasticity of substitution of Andreas' demand for good 1 is zero.

The inverse elasticity rule tells that it is optimal to tax more heavily goods that are demanded less elastically.

Here we have the extreme case. In fact, with Leontief preferences, taxation has no efficiency costs in collecting taxes from individuals with Andrea's preferences (the deadweight loss of taxes is zero).

<u>Part B (15%):</u>

A firm (in a standard price taking setting) "Nut" produces with the technology:

$$y \leq f_n(z_1, z_2) = (z_1 z_2)^{\frac{1}{4}} .$$

B1 (8%): Derive the conditional demand for inputs and the cost function of Nut.

SB1: The cost minimization problem is:

$$\min_{z_1, z_2} w_1 z_1 + w_2 z_2 \ s.t. \ y \leq (z_1 z_2)^{\frac{1}{4}} .$$

The Lagrangean is

$$\mathcal{L}(z_1, z_2, \lambda; w_1, w_2, y) = w_1 z_1 + w_2 z_2 - \lambda \left((z_1 z_2)^{\frac{1}{4}} - y \right) .$$

The solution is interior for each $y > 0$ (with $z_i = 0$ production is 0, violating the constraint). The FOCs are:

$$\begin{cases} \lambda^* \frac{\partial f(z_1^*, z_2^*)}{\partial z_1} \leq w_1 \\ \lambda^* \frac{\partial f(z_1^*, z_2^*)}{\partial z_2} \leq w_2 \\ y \leq (z_1^* z_2^*)^{\frac{1}{3}} \end{cases}$$

which, substituting, give:

$$\begin{cases} \lambda^* \frac{1}{3} (z_1^*)^{-\frac{3}{4}} (z_2^*)^{\frac{1}{4}} = w_1 \\ \lambda^* \frac{1}{3} (z_1^*)^{\frac{1}{4}} (z_2^*)^{-\frac{3}{4}} = w_2 \\ y = (z_1^* z_2^*)^{\frac{1}{3}} \end{cases}$$

Dividing the first by the second FOC (and taking the 4th power of the third one), gives:

$$\begin{cases} \frac{z_2^*}{z_1^*} = \frac{w_1}{w_2} \\ y^4 = z_1^* z_2^* \end{cases}$$

and, solving for z_2^* gives:

$$z_2^* = \frac{w_1}{w_2} z_1^* = \frac{w_1}{w_2} \frac{y^4}{z_2^*}$$

Thus:

$$(z_2^*)^2 = y^4 \frac{w_1}{w_2}$$

and the conditional demand function of input 2 is:

$$z_2^* = z_2^h (w_1, w_2, y) = y^2 \sqrt{\frac{w_1}{w_2}}$$

Substituting back for the conditional demand function of input 1 gives:

$$z_1^* = z_1^h (w_1, w_2, y) = y^2 \sqrt{\frac{w_2}{w_1}}.$$

The cost function is then

$$c (w_1, w_2, y) = w_1 z_1^h (w_1, w_2, y) + w_2 z_2^h (w_1, w_2, y) =$$

$$= w_1 y^2 \sqrt{\frac{w_2}{w_1}} + w_2 y^2 \sqrt{\frac{w_1}{w_2}} = 2y^2 \sqrt{w_1 w_2}.$$

B2 (7%):Find the profit function for this firm.

SB1: The profit maximization problem is

$$\max_y py - c (w_1, w_2, y) = \max_y py - 2y^2 \sqrt{w_1 w_2},$$

with FOC

$$p = 4y^* \sqrt{w_1 w_2},$$

and (since SOC is $-4\sqrt{w_1 w_2} < 0$*) optimal supply:*

$$y^* = y(p, w_1, w_2) = \frac{p}{4\sqrt{w_1 w_2}}.$$

The profit function is

$$\pi(p, w_1, w_2) = py^* - 2(y^*)^2 \sqrt{w_1 w_2}$$

$$= \frac{p^2}{4\sqrt{w_1 w_2}} - 2\frac{p^2}{16 w_1 w_2}\sqrt{w_1 w_2} = \frac{p^2}{4}(w_1 w_2)^{-\frac{1}{2}} - \frac{p^2}{8}(w_1 w_2)^{-\frac{1}{2}} =$$

$$\pi(p, w_1, w_2) = \frac{p^2}{8}(w_1 w_2)^{-\frac{1}{2}}.$$

Part C (15%):

Consider an economy where the only agents are:

(Andrea's cousin) Carl, with preferences $U(x_1, x_2) = \min[x_1, x_2]$ and an endowment vector $(\omega_1, \omega_2) = (0, 2)$; and

the firm "Bolt" that produces with the technology:

$$y \leq f_b(z_1) = z_1^2.$$

Good 1 (x_1) is the same as the output y of the firm. Good 2 (x_2) is the same as the input z_1 of the firm.

C1 (4%): Discuss the profit maximization problem of the firm Bolt.

SC1: Bolt has increasing returns to scale. For all cases of increasing returns to scale, the optimum is to either produce 0 or ∞ *(the second order condition for an interior solution is violated). In this specific case, the profit maximization problem is:*

$$max_y py - w_1\sqrt{y}.$$

For all $p, w_1 > 0$*, producing* $y = \infty$ *gives a profit* $py - w_1\sqrt{y} > 0$*. Thus,* $y = 0$ *is never optimal.*

C2 (4%): Determine the set of feasible allocations for this economy.

SC2: The feasibility constraints are $x_1 \leq y$ *and* $x_2 \leq 2 - z_1$ *(with all non-negative quantities) and, since* $y \leq z_1^2$*, we can write the feasibility constraints as* $x_1 \leq z_1^2$ *and* $x_2 \leq 2 - z_1$*.*

C3 (4%): Argue that the allocation of consumption $(x_1^*, x_2^*) = (1, 1)$ and of production $(y^*, z_1^*) = (1, 1)$ is Pareto efficient.

SC3: Pareto efficient here means that Carl cannot be better off. At an optimum for Carl, he consumes equal quantities of goods. Thus, $x_1 = x_2 = z_1^2 = 2 - z_1$*. It is immediate that* $z_1 = 1$ *is the only positive level of input that satisfies this constraint. With* $z_1 = 1$*, output is* $y = z_1^2 = 1$ *and consumptions are* $x_1 = y = 1$ *and* $x_2 = 2 - z_1 = 2 - 1 = 1$*.*

C4 (3%): Can this allocation be a Walrasian equilibrium allocation? Briefly justify your answer (Hint. You can use two arguments: one based on the profit maximization problem of the firm; and one based on the assumptions of the second welfare theorem).

SC4: The second welfare theorem requires that preferences and technology are convex. Here, convexity is violated for the firm.

In fact, we know that the profit maximization problem for the firm has no interior solution. Thus, there exist no prices for which the price-taking firm would optimally produce $y^* = 1$ *with inputs* $z_1^* = 1$*.*