

Exam

Question 1 (10%)

Define and explain the following terms: auction, pure strategy, mixed strategy, Nash equilibrium.

Solution: See Watson or Varian & Melitz.

Question 2 (20%)

Two agents engage in activities to influence the outcome of a decision process (think, for example, of the allocation of a right to exploit a natural resource). The effort exerted by agent i is denoted $e_i \geq 0, i = 1, 2$. The agents decide simultaneously how much effort to exert. The agent exerting the most effort succeeds in tilting the outcome of the process to his or her advantage; specifically, if $e_1 > e_2$, agent 1 receives the value $v_1 > 0$, while agent 2 receives zero; correspondingly, if $e_2 > e_1$, agent 2 receives the value $v_2 > 0$, while agent 1 receives zero; if $e_1 = e_2$, both agents receive zero. Values and efforts are measured in the same unit.

A

Explain why the payoff to player i is $v_i - e_i$ if $e_i > e_j$ and $-e_i$ if $e_i \leq e_j, i, j = 1, 2, i \neq j$.

Solution: Straightforward.

B

Explain why this set up (or game) may be analysed as an all-pay auction.

Solution: An all-pay auction is an auction in which every bidder must pay their bid regardless of whether they win the prize, which is awarded to the bidder with the highest bid.

Question 3 (40%)

We continue with the set up presented above, and suppose it is common knowledge that $v_1 = v_2 = v$.

A

Explain why there does not exist a Nash equilibrium in pure strategies.

Solution: Clearly there cannot be an equilibrium in which an agent i chooses $e_i > v$, since then this bidder would end up with a negative payoff. Moreover, if $e_1 \leq e_2 < v$, agent 1 would benefit from choosing $e_2 < e_1 < v$, while, if $0 < e \leq e_2 \leq v$, agent 1 would benefit from reducing effort.

B

Explain that in any mixed-strategy equilibrium the expected payoff to each agent must be zero (hint: consider the expected payoff at the lower bound of the support of the mixed strategies).

Solution: Clearly, the expected payoff at equilibrium must be greater than or equal to zero, since otherwise agents would not exert effort. Moreover, at the lower bound of the support of the mixed strategies, $\underline{e}$, the probability of winning is zero and the payoff is $-\underline{e} \leq 0$, implying

that we must have $\underline{e} = 0$. Since agents must be indifferent between any action in the support of the mixed strategies, the expected payoff is zero.

C

Consider a symmetric mixed-strategy equilibrium in which agent $i, i = 1, 2$, chooses e_i according to the probability function (or cumulative distribution function) $F(e)$, with $F(0) = 0$, $F(v) = 1$ and $F'(e) > 0$ for all $e \in [0, v]$. Demonstrate that the expected payoff from exerting effort e is $F(e)v - e$, that $F(e) = \frac{e}{v}$ (i.e. F is uniform) and that $Ee = \frac{v}{2}$. What is the expected total amount of effort spent? Explain.

Solution: The expected payoff is $F(e)(v - e) - (1 - F(e))e = F(e)v - e$. From the condition $F(e)v - e = 0$, the second result follows. We have $Ee = \int_0^v e \frac{1}{v} de = \frac{v}{2}$. The expected total amount of effort exerted is $2Ee = v$; i.e. all gains are dissipated. Put differently, since the gain to each player is zero (and the probability of a tie is zero), effort must equal value.

D

Suppose the amount of effort exerted by any agent is capped, i.e. $e_i \leq \bar{e}, i = 1, 2$. Discuss how the cap affects the outcome. Restricting attention to the two agents, does the cap increase efficiency?

Solution: The cap does not affect equilibrium for efforts below the cap, but efforts will be limited by the cap (effort at the cap will be played with positive probability). Both players earn a payoff of zero (including at the cap), hence efficiency is not affected.

Question 4 (30%)

Again in the same set up as above, suppose now that it is common knowledge that the values v_1 and v_2 are independently and uniformly distributed between zero and v .

A

Consider a symmetric equilibrium in which agent i spends effort according to the function $e_i = e(v_i), i = 1, 2$. Demonstrate that the expected payoff to agent i is given by $Pr\{v_j \leq e^{-1}(e_i)\}v_i - e_i, i, j = 1, 2, j \neq i$ (where e^{-1} is the inverse function of e) and show that $e(v_i) = \frac{1}{2v}v_i^2$. What is the expected total amount of effort spent? Explain.

Solution: The expected payoff is $Pr\{e(v_j) \leq e_i\}(v_i - e_i) - (1 - Pr\{e(v_j) \leq e_i\})e_i = Pr\{e(v_j) \leq e_i\}v_i - e_i$. Since $Pr\{e(v_j) \leq e_i\} = Pr\{v_j \leq e^{-1}(e_i)\}$, the first result follows. Maximising the latter expected payoff with respect to e_i , and using that $e^{-1'}(e_i) = \frac{1}{e'(v_i)}$, we find $e(v_i) = \frac{1}{2v}v_i^2 + A$. Then using that we must have $e(0) = 0$, the second result follows. Finally, $Ee = \int_0^v \frac{1}{2v}v_i^2 \frac{1}{v} dv_i = \frac{1}{6}v$ and so the expected total amount of effort spent is $2Ee = \frac{1}{3}v$. So, in this case, not all gains are dissipated.

B

Suppose again that the amount of effort exerted by any agent is capped, i.e. $e_i \leq \bar{e}, i = 1, 2$. Discuss how the cap affects the outcome. Does the cap increase efficiency?

Solution: Again, bidding functions for efforts below the cap remain the same; for higher values efforts will equal the cap. Since efforts are reduced for higher values, total expected efforts is

reduced. Values are also reduced, since ties (at the cap) lead to no decision; in particular, the decision does not favour the agent with the higher value.