Expected utility and the independence axiom
A simple exposition of the main ideas

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1 Introduction

Expected utility is a theory on how we choose between lotteries. It can be seen as only a normative theory about how we ought to choose or a positive theory that predicts how people actually choose. Those are not the same. It is possible to hold a position that my choices are not like this (reject the positive theory) but still think that I wish I was that consistent (accept it as a normative theory).

1.1 Lotteries:

Let us first describe what we mean by a lottery. A lottery is a set of outcome each assigned a probability such that the joint probability sums to one. The outcome can be anything: "Linda falls in love with me", "I get an A on the final exam". For simplicity we will here limit the attention to monetary outcome: "I earn 100 kroner", "I loose 37 kroner" etc. Note though that the logic behind expected utility works for any kind of lottery. To simplify even further I assume that all outcomes are in the range -1000 to +1000 kroner, and that we consider a person who wants more money rather than less.

A typical lottery is then like this:
Win 100 kroner with probability 43%
Win 214 kroner with probability 34%
Loose 23 kroner with probability 23%
Note:

- There can be as many outcomes as you like but there must be at least one
- The probabilities has to sum to 100%

If there is only one outcome this has to get 100%, thus to get 23 kroner for sure is a lottery.

More generally, a lottery is a list of outcomes x_i and associated probabilities p_i, we write it $(x_1, p_1; x_2, p_2; ....; x_n, p_n)$ The outcome x_i has probability p_i and probabilities add to one: $\sum p_i = 1$.

1.2 Expected Utility

We want to rank lotteries so lets look at two lotteries

$$\begin{aligned} L_1 &= (x_1, p_1; x_2, p_2; ....; x_n, p_n) \text{ (With } n \text{ different outcomes)} \\ L_2 &= (y_1, q_1; y_2, q_2; ....; y_n, q_n) \text{ (With } n \text{ different outcomes)} \end{aligned}$$

Next let u be a utility function with outcomes as argument. Note that this is different from standard utility fucntions in the sense that if we rank baskets of apples and oranges, the argument will be baskets of apples and oranges. Now we rank lotteries but the argument is outcomes.

Now consider a person who get lottert L_1. Depending on the outcome of the lottery this person may end up with utility $u(x_1)$ or perhaps $u(x_2)$ or ... $u(x_n)$. We cannot know what utility the person get, but we can determine expected utility.

$$EU = \sum_{i=1}^{n} p_i u(x_i)$$

Claim 1 *(Expected utility theory): Lotteries are ranked such that*

$$L_1 \succ L_2 \Longleftrightarrow \sum_{i=1}^{n} p_i u(x_i) > \sum_{i=1}^{n} q_i u(y_i)$$

1.2.1 Why we can choose $u(0) = 0$

Note first that the function $v(x) = u(x) + c$ for some constant term c will give the same ranking, as

$$\begin{aligned} \sum_{i=1}^{n} p_i v(x_i) &> \sum_{i=1}^{n} q_i v(y_i) \\ &\Longleftrightarrow \sum_{i=1}^{n} p_i u(x_i) + c > \sum_{i=1}^{n} q_i u(y_i) + c \\ &\Longleftrightarrow \sum_{i=1}^{n} p_i u(x_i) > \sum_{i=1}^{n} q_i u(y_i) \end{aligned}$$

As a consequence we, if started with a utility function $\tilde{u}$ such that $\tilde{u}(0) \neq 0$ we simply add the constant $c = -\tilde{u}(0)$, and use $u(x) = \tilde{u}(x) + c$, and then $u(0) = 0$. Thus we can always assume that $u(0) = 0$.

1.2.2 An illustration that expected utility is not something obvious

Consider the lotteries

A: winning 300 with 100% probability
B: Winning 400 with 80% probability.
C: winning 300 with 25% probability
B: Winning 400 with 20% probability.

Theorem 2 *If preferences satisfies Expected Utility then*

$$A \succ B \Longleftrightarrow C \succ D$$

Note the difference to standard utility theory: If you prefer 100 oranges to 80 apples you can still prefer 20 apples to 30 oranges.

1.3 Axioms

I am not going to give a complete formal proof of expected utility here, only to convey the basic idea. What we want to show is that if a person's preferences over lotteries have certain properties - axioms - they will be of the expected utility form. You will not be asked to reproduce any proof on exam, but it will help to understand one of the behavioral theories if you grasp the importance of the independence axiom.

As you are used to elsewhere we assume that preferences are complete (subjects can compare any two lotteries) and transitive (If lottery A is better than B and B better than C then A is better than C). But we also require some more, and before we get to independence we need technical axiom:

1.3.1 Continuity

Consider two lotteries.

A: You get x kroner with certainty

B: You get 1000 kroner with probability p and -1000 kroner with probability $1 - p$

For what value of p will you be indifferent between A and B?

Axiom 3 *For any lottery A there will be a value p such that $A \sim B$*

Some notation; let x be the amount you get in A, and $u(x)$ the probability such that you are indifferent between A and B. For later reference, let us call this lottery $L(x)$. That is

$L(x)$: You get 1000 kroner with probability $u(x)$ and -1000 kroner with probability $1 - u(x)$

1.4 The independent axiom

To avoid having to introduce more notation I will not state the full axiom, but the following should give you and idea. Suppose you are indifferent between lotteries A and B. The independence axiom states that this indifference should be independent of context. That is if you put A and B inside another lottery you are still indifferent.

Lottery C: You win lottery A with probability q

Lottery D: You win lottery B with probability q

Since you are indifferent between A and B you should also be indifferent between C and D. The following is a sloppy formulation, but let's call it an axiom still.

Axiom 4 *If you are indifferent between A and B you will still be so if A and B appear inside a lottery.*

1.5 Getting rid of intermediate outcome.

Now consider the lottery

Lottery X : x with probability p, and y with probability $1-p$.

We know that the person will be indifferent between x and the lottery $L(x)$. Thus if we replace x in the lottery above with $L(x)$,

X' : Get the lottery $L(x)$ with probability p and y with probability $1-p$.

By the independence axiom $X \sim X'$. Now we have a lottery X' where we get either -1000, +1000 or y. Next we get rid of y. We know the person is indifferent between y for sure or the lottery $L(y)$.We thus replace y with $L(y)$ to get.

X'' : Get the lottery $L(x)$with probability p and $L(y)$ with probability $1-p$

Now by independence $X \sim X' \sim X''$ and note finally that the last lottery can be written

X'' :Get 1000 with probability $u(x)p + u(y)(1-p)$ and -1000 with the remaining probability.

Now the point is that the winning probability $u(x)p+u(y)(1-p)$ is exactly the expected value of the function u, that is expected utility.

1.5.1 With a general lottery

We can do the same with any lottery $L = (x_1, p_1; x_2, p_2; ....; x_n, p_n)$ by replacing each outcome x_i with the lottery L_i where you get 1000 with probability $u(x_i)$ and -1000 with the remaining probability, we get

$$L \sim X$$

where

$$X = (1000, P; -1000, 1-P) \text{ where } P = \sum_{i=1}^{n} p_i u(x_i)$$

Since we want the winning probability P to be high we maximize expected utility.