

Problems to be discussed at the 5th seminar

Suggested solutions

Problem 1

- a) Consider an ultimatum game in which the proposer gets, initially, 100 NOK. Assume that both the proposer (A) and the responder (B) have inequity aversion as specified in the model of Fehr and Schmidt (1999), with $\alpha_i = 3$ and $\beta_i = 0.3$, $i = A, B$. Explain the optimal strategy of the two players if these preferences are common knowledge.

In the 2-player version of the FS (1999) model, preferences are specified as

$$U_i = x_i - \alpha_i \max\{x_j - x_i, 0\} - \beta_i \max\{x_i - x_j, 0\}$$

where $i \neq j$, and $\beta_i \leq \alpha_i$, $0 \leq \beta_i < 1$. (i prefers that j 's income is equal to hers; i 's utility declines in their income *difference*, more so if i herself is worst off.) Using the parameter values in 1a), we get that

$$U_i = x_i - 3 \max\{x_j - x_i, 0\} - 0.3 \max\{x_i - x_j, 0\}$$

Utility consists of material payoff minus a loss from inequity. A rejected offer thus always implies a utility of **zero** for both players (in this case, the outcome is symmetrical): no material payoff and no inequity.

$$U_i = 0 - 3 \max\{0 - 0, 0\} - 0.3 \max\{0 - 0, 0\} = 0$$

Consider first the Responder's choice.

Any offers $s \geq 0.5$ will be accepted:

If $s=0.5$, and the offer is accepted, there is no loss from inequity, but material payoff is strictly positive. Hence the offer is accepted. (This could of course be shown formally - take a look at the utility function to check that the claim makes sense.)

If $s > 0.5$, the Responder's utility is

$$U_B = x_B - 0.3(x_B - x_A) = x_B - 0.3x_B + 0.3x_A = 0.7x_B + 0.3x_A$$

which is strictly positive regardless of how the 100 NOK are shared, i.e. larger than the utility of Reject, so offer is accepted. (Alternatively, you can solve w.r.t s , just like below.)

To decide whether offers **below 0.5** will be accepted, we must compare the Responder's utility of rejecting – which is 0 – to the case in which A gets $X(1-s)$ and B gets sX . If the responder accepts, his utility will be (using that $s < 0.5$, so i gets more than j)

$$\begin{aligned} U_i &= 100s - 3 \max\{100(1-s) - 100s, 0\} - 0.3 \max\{100s - 100(1-s), 0\} \\ &= 100s - 3[(1-s)100 - 100s] \\ &= 100(s - 3 + 3s + 3s) \\ &= 100(7s - 3) \end{aligned}$$

This is less than 0, and thus worse than reject, if

$$100(7s - 3) < 0$$

i.e.

$$7s < 3$$

$$s < 3/7$$

Hence, the Responder will accept anything above 3/7 (and is indifferent between Reject and Accept when $s = 3/7$).

Consider then the Proposer (A).

Since $\beta_A = 0.3 < 0.5$, he will prefer to keep everything for himself (see notes to lecture 2).

But since he knows that B will reject if $s < 3/7$, he will offer exactly 3/7 (or just *slightly* more to ensure strict preference), and B will accept.

This is the quick&easy answer, which is a little unsatisfactory because I just refer to a result from the literature without explaining it. Your answer will be better if you show that a person with $\beta_A = 0.3$ will prefer to keep everything to himself, or explain intuitively why, in general, A will prefer to keep everything when $\beta_A < 0.5$.

Formally: Consider the case where $s < 0.5$ (a similar analysis can be done for $s \geq 0.5$). Then

$$\begin{aligned} U_A &= 100(1-s) - 0.3(100(1-s) - 100s) \\ &= 100(1-s - 0.3(1-s-s)) = 100(1-s - 0.3 + 0.6s) \\ &= 100(0.7 - 0.4s) \end{aligned}$$

Then one can see that A's utility is decreasing in s :

$$\partial U_A / \partial s = -0.4.$$

Thus, A will prefer s as low as possible.

Why, in general, will a person with $\beta_i < 0.5$ prefer to keep everything to himself?

Intuitive explanation: When A gives B one kr, A loses one kr, but inequity is changed by 2 kr (A gets 1 kr poorer and B gets one kr richer). Thus, to want to give money away, A must place a weight on inequity which is at least half as big as the weight he places on income. (When $s < 0.5$, A is richest, so this reasoning means that $\beta_i \geq 0.5$. If $s > 0.5$, B is richest and α_i is the relevant parameter; but we know that $\alpha_i \geq \beta_i$ – that's an assumption of the model, so if beta isn't big enough, alpha won't be either.)

(Or: When A is better off, the utility function can be written

$$U_A = x_A - \beta_A (x_A - x_B)$$

To see what happens if A gives 1 kr to B, we can differentiate and use $-1 = dx_A = -dx_B$:

$$dU_A = dx_A - \beta_A (dx_A - dx_B) = -1 - \beta_A (-1 - 1) = -1 + 2\beta_A$$

A is exactly indifferent when $(-1 + 2\beta_A) = 0$, i.e. $\beta_A = 1/2$.)

- b) Assume that the responder (B) has the preferences specified in Question 1a), but that the proposer (A) is only concerned about his own material payoff. What are now the players' optimal strategies, if these preferences are common knowledge?

The answer is exactly like in 1a). The Proposer's aversion against advantageous inequity in question 1a) is insufficient to make him want to share. He shares only because B would otherwise reject. This corresponds exactly to the behavior of a self-interested Proposer.

- c) Assume that the proposer (A) has the preferences specified in Question 1a), but that the responder (B) is only concerned about his own material payoff. What

is now the optimal strategy for the two players, assuming that their preferences are common knowledge?

The Proposer still does not want to share, as in 1a). The Responder now does not have any credible threat to reject strictly positive amounts, since the Responder knows his preferences. Hence, predictions are exactly like in the self-interest case (see lecture notes, Lecture 1 of Topic 3): If the lowest possible amount is 50 øre, there are two subgame perfect Nash equilibria: One in which the proposer proposes 50 øre and the responder accepts any strictly positive offer, and another in which the proposer proposes nothing and the responder accepts anything.

Problem 2

Consider the following game:

	<i>Cooperate</i>	<i>Defect</i>
<i>Cooperate</i>	3,3	-5, 5
<i>Defect</i>	5,-5	0,0

The rows correspond to possible choices of Player A, while columns correspond to possible choices of Player B. The first number in each cell denotes player A's material payoff, the second number in each cell denotes B's material payoff.

Think of A and B as partners in a firm. If both invest 10 in a project, the project will achieve an income of 13 (per person), so both will get net earnings of 3. If only one of them invests, the project earns only 5 (per person), leading to a payoff of -5 for the person who invested and 5 for the other. If none of them invests, both get nothing.

- a) *What are the players' dominant strategies if each cares only about his own material payoff?*

This is a Prisoners' Dilemma game, so Defect (do not invest) is a dominant strategy for both players: Consider the position of player A. If player B cooperates (invests), A gets 3 if he invests too, and 5 if he defects (does not invest). If player B defects (does not invest), A gets -5 if he invests, and 0 if he does not invest. Hence, regardless of what B does, A is better off by not investing, so Defect (do not invest) is A's dominant strategy. By symmetry, the same holds for B. (So (Defect, Defect) is the only Nash equilibrium of this game.)

- b) *Assume that both players have inequity aversion as specified in Question 1a). Is (Cooperate, Cooperate) then a Nash Equilibrium?*

Assume first that player B cooperates (invests). Then, if A cooperates (invests) too, both get a material payoff of 3, so A's utility is

$$U_i = x_i - 3 \max\{x_j - x_i, 0\} - 0.3 \max\{x_i - x_j, 0\} \\ = 3 - 0 = 3$$

(since there is no inequity). The same holds for B (his utility is also 3).

Similarly, if both defect, there is no inequity, and each player's utility equals his material payoff, namely 0.

If A defects and B cooperates, A's utility is

$$U_A = 5 - 3 \cdot 0 - 0.3(5 - (-5)) \\ = 5 - 0.3 \cdot 10 = 2$$

which is less than his material payoff because he dislikes disadvantageous inequity.

(If A cooperates and B defects, A's payoff is

$$U_A = -5 - 3(5 - (-5)) - 0.3 \cdot 0 \\ = -5 - 3 \cdot 10 = -35$$

but this is not really needed to arrive at the answer that C,C is a Nash eq.).

Assume now that both invest (cooperate). Then A's utility is 3. If he changes his strategy and defects, while B's strategy is kept fixed, A's utility is 2, which is lower than 3. By symmetry, the same reasoning holds for B. Consequently, (Cooperate, Cooperate) is a Nash equilibrium with these preferences.

You can see this clearly by writing down a payoff matrix in *utilities* rather than *money*. Although some numbers in this matrix are, strictly speaking, not needed to arrive at the answer, this is a nice way to provide an overview.

	<i>Cooperate</i>	<i>Defect</i>
<i>Cooperate</i>	3,3	-35, 2
<i>Defect</i>	2,-35	0,0

It is now straightforward to verify that C is each player's best response to the other playing C. (D,D) is also a Nash equilibrium.

- c) Assume that player A has inequity aversion as specified in Question 1a), but that player B cares only about his own material payoff. Is (Cooperate, Cooperate) then a Nash Equilibrium?

A's payoffs are as derived in question b) above. However, B cares only about his material payoff, which is 3 in the case of mutual cooperation and 5 if he defects and A cooperates. Hence, (C,C) cannot be a Nash equilibrium: Compared to this situation, player B can gain by changing his strategy to Deviate.

Utilities are now

	<i>Cooperate</i>	<i>Defect</i>
<i>Cooperate</i>	3,3	-35, 5
<i>Defect</i>	2,-5	0,0

so D,D is now the only Nash equilibrium.

Problem 3

Assume now that both players in the game described in Problem 2 above have reciprocal preferences. Let player i 's utility U_i be defined as follows:

$$U_i = x_i + k_{ij} \tilde{k}_{ji}$$

where x_i = i 's material payoff, k_{ij} = i 's kindness towards j , and \tilde{k}_{ji} = i 's belief about j 's kindness towards i ($i=1,2$; $j=1,2$; $i \neq j$).

3 a) Does it seem reasonable that a person with reciprocal preferences as specified above might prefer to play Cooperate in this game, if he expects that the other will play Cooperate? Explain the main intuition (you should not necessarily have to define "kindness" formally to be able to do this).

If the other is "kind", he wants to be kind in return; if the other is "mean", he wants to be "mean" in return. It seems intuitively reasonable to classify "cooperate" as a kind action, triggering a desire to be kind in return. If this reciprocal preference is strong enough, it may outweigh the concern for material payoff (in a fairly similar fashion as in the inequity aversion example in Seminar 5).

(Note: The intuitive reasoning that i perceives j as kind if j Cooperates because j is sacrificing some of his material payoff to help i , is perfectly ok. But note that this particular idea is not captured in the formal definition of kindness below, since that definition does not include j 's payoff (his sacrifice) in the measurement of j 's kindness, only the payoff that j "secures" to i .)

3 b) Explain what a fairness equilibrium is.

A fairness equilibrium is a situation in which every player maximizes his utility, given his beliefs and the other players' strategies, and where beliefs are correct. Thus, in a fairness equilibrium, no player has any reason to change his strategy and/or his beliefs, given the strategies and beliefs of the other players.

In standard game theory, it is assumed that players' payoffs (utility) depend only on outcomes, not on beliefs *per se*. In models of reciprocity, utility can depend directly on beliefs, not just on outcomes (e.g.; I may feel bad if I believe that your intentions were mean, even if I'm wrong), thus violating this assumption from standard game theory. The concept of fairness equilibrium is quite similar to a Nash equilibrium, but is developed for models in which beliefs may matter directly for payoffs.

Let us now define "kindness" formally. Assume that i 's kindness towards j is defined in the following way:

$$k_{ij} = x_j(s_i, b_{ij}) - 1/2[x_j^{\max}(b_{ij}) + x_j^{\min}(b_{ij})]$$

where s_i = i 's strategy (Cooperate, or Defect),

b_{ij} = i 's belief about j 's strategy (Cooperate, or Defect),

$x_j^{\max}(b_{ij})$ is the largest material payoff i could secure to j , given i 's belief about j 's strategy b_{ij} , and $x_j^{\min}(b_{ij})$ is the smallest material payoff i could secure to j , given b_{ij} .

Hence, kindness is given by the payoff i allocates to j compared to the average of those payoffs i could potentially have allocated to j (given his beliefs).

Moreover, let c_{iji} = i 's belief about j 's belief about i 's strategy. We can then define \tilde{k}_{ji} in the same way as k_{ij} , but taking into account that to evaluate j 's kindness towards himself (i), i must use his beliefs about j 's strategy, b_{ij} , and his beliefs about j 's beliefs about i 's own strategy (c_{iji}):

$$x_i(b_{ij}, c_{iji}) - 1/2[x_i^{\max}(c_{iji}) + x_i^{\min}(c_{iji})]$$

3 c) What is/are the fairness equilibrium/equilibria in the game described above, given the reciprocal preferences and definitions of kindness specified here? Explain.

Only states where beliefs are correct can be candidates for fairness equilibria.

For a state to be a fairness equilibrium, no player can gain utility by *unilaterally* deviating from it. Unilateral deviation from a fairness eq. implies doing something else than your opponent expects. Thus, to check whether the candidate is indeed a fairness eq., we need to compare it with the alternatives for unilateral deviation, which are NOT fairness equilibria.

Let us first calculate each person's utility in each of the potential fairness eq. To calculate utilities, we must first calculate kindness and evaluations of the other's kindness.

Is (C,C) a fairness equilibrium?

Consider person 1's perspective. Assume that 1 thinks 2 will play Cooperate ($b_{12}=C$) and that 1 thinks 2 believes 1 will play Cooperate too ($c_{121}=C$). Then, 2's kindness to 1, as evaluated by 1, is as follows:

$$\begin{aligned}\tilde{k}_{21} &= x_1(b_{12}, c_{121}) - 1/2[x_1^{\max}(c_{121}) + x_1^{\min}(c_{121})] \\ &= x_1(C, C) - 1/2[x_1^{\max}(C) + x_1^{\min}(C)] \\ &= 3 - 1/2[3 + (-5)] = 3 - (-2/2) = 3 + 1 = 4\end{aligned}$$

Note that 1's actual strategy s_1 does not enter the expression for \tilde{k}_{21} , only 1's beliefs.

When we calculate 1's actual kindness towards 2, k_{12} , actual strategy s_1 matters. If 1 keeps to his (possible) equilibrium behaviour, i.e. $s_1 = C$, then (by symmetry) we must have,

$k_{12} = \tilde{k}_{21} = 4$. Inserting this in the utility function, and using that the game is symmetric for 1 and 2, we get that the utility of player $i=1,2$ in (C, C) is

$$U_i = x_i + k_{ij}\tilde{k}_{ji} = 3 + 4 \cdot 4 = 19$$

Unilateral deviation from a (possible) fairness eq. (C, C) for player 1 means to play D even though 2 expects him to play C. That is, $s_1=D$, $b_{12}=C$, $c_{121}=C$. In this case,

$$\begin{aligned}k_{12} &= x_2(s_1, b_{12}) - 1/2[x_2^{\max}(b_{12}) + x_2^{\min}(b_{12})] \\ &= x_2(D, C) - 1/2[x_2^{\max}(C) + x_2^{\min}(C)] \\ &= -5 - 1/2[3 + (-5)] = -5 + 1 = -4\end{aligned}$$

Inserting this in the utility function, we find 1's utility by unilateral deviation from (C,C), that is $s_1=D$, $b_{12}=C$, $c_{121}=C$:

$$U_1 = x_1 + k_{12}\tilde{k}_{21} = 5 - 4 \cdot 4 = -11$$

Thus, deviation from (C,C) decreases utility for 1. By symmetry the same holds for 2. Hence (C,C) is a fairness equilibrium.

Is (D,D) a fairness equilibrium?

If both play D, and believe that the other plays D, a similar reasoning gives

$$\begin{aligned}\tilde{k}_{21} &= x_1(b_{12}, c_{121}) - 1/2[x_1^{\max}(c_{121}) + x_1^{\min}(c_{121})] \\ &= x_1(D, D) - 1/2[x_1^{\max}(D) + x_1^{\min}(D)] \\ &= 0 - 1/2[5+0] = -(5/2)\end{aligned}$$

and again $\tilde{k}_{21} = k_{12}$ by symmetry. Hence, if both play D and believe that the other will play D, the utility of each player $i=1,2$ is

$$\begin{aligned}U_i &= x_i + k_{ij}\tilde{k}_{ji} \\ &= 0 + (-5/2)(-5/2) = 25/4 = 6.25.\end{aligned}$$

This is a fairness eq. if neither 1 nor 2 can gain from unilateral deviation from (D,D). If 1 deviates and plays C, while 2 still thinks 1 will play D, $\tilde{k}_{21} = -(5/2)$ as before, while k_{12} is given by

$$\begin{aligned}k_{12} &= x_2(s_1, b_{12}) - 1/2[x_2^{\max}(b_{12}) + x_2^{\min}(b_{12})] \\ &= x_2(C, D) - 1/2[x_2^{\max}(D) + x_2^{\min}(D)] \\ &= 5 - 1/2[5+0] = 5 - 5/2 = 5/2\end{aligned}$$

Inserting this into the utility function gives

$$U_1 = -5 + (-5/2)(5/2) = -11.25.$$

By symmetry of the game the same must hold for 2. Hence (D,D) is a fairness eq.

Is (C,D) a fairness equilibrium?

Assume that 1 thinks 2 will play Defect ($b_{12}=D$) and that 1 thinks 2 believes 1 will play Cooperate ($c_{121} = C$). Then 1's evaluation of 2's kindness gives

$$\begin{aligned}\tilde{k}_{21} &= x_1(b_{12}, c_{121}) - 1/2[x_1^{\max}(c_{121}) + x_1^{\min}(c_{121})] \\ &= x_1(D, C) - 1/2[x_1^{\max}(C) + x_1^{\min}(C)] \\ &= -5 - 1/2[3 + (-5)] = -5 - (-2/2) = -5 + 1 = -4\end{aligned}$$

What is 1's actual kindness towards 2? If he plays C (yielding material payoffs $(-5,5)$), we get

$$\begin{aligned}k_{12} &= x_2(s_1, b_{12}) - 1/2[x_2^{\max}(b_{12}) + x_2^{\min}(b_{12})] \\ &= x_2(C, D) - 1/2[x_2^{\max}(D) + x_2^{\min}(D)] \\ &= 5 - 1/2[5+0] = 5 - 5/2 = 5/2\end{aligned}$$

and his utility will be

$$U_1 = x_1 + k_{12}\tilde{k}_{21} = -5 + \frac{5}{2} \cdot (-4) = -5 - 10 = -15$$

If 1 instead plays D, while expectations are unchanged, his kindness is

$$k_{12} = x_2(s_1, b_{12}) - 1/2[x_2^{\max}(b_{12}) + x_2^{\min}(b_{12})]$$

$$\begin{aligned}
&= x_2(D, D) - 1/2[x_2^{\max}(D) + x_2^{\min}(D)] \\
&= 0 - 1/2[5+0] = 0 - 5/2 = -5/2
\end{aligned}$$

and the utility of player 1 is

$$U_1 = x_1 + k_{12}\tilde{k}_{21} = 0 + \left(-\frac{5}{2}\right) \cdot (-4) = 0 + 10 = 10$$

Hence (C,D) is NOT a fairness equilibrium.

Note: I make this conclusion because *at least one player* (player 1) can benefit from deviating. Thus it's not necessary to calculate utilities for Player 2. Had we found that Player 1 could *not* benefit from deviating, we would need to check if the same was true for Player 2, because the situation (C,D) is NOT symmetrical.

Is (D,C) a fairness equilibrium?

NO. Short answer, see discussion of (C,D) above for explanation:

$$b_{12}=C, c_{121}=D$$

$$\tilde{k}_{21}=2.5$$

$$\text{If } s_1=D: k_{12}=-4, U_1=-5$$

$$\text{If } s_1=C: k_{12}=4, U_1=13$$

Summary – not a game matrix

We may well summarize the payoffs in utilities in all situations with correct beliefs, as follows:

	C	D
C	19,19	-15,-5
D	-5,-15	6.25,6.25

However, this should NOT be regarded as a fully specified game matrix. The reason is that some alternatives, namely those choices implying that a player thinks he will surprise the other, are not included in the table. Thus we *cannot* in general find fairness equilibria by using the above table as a standard game and looking for its Nash equilibria (although this would produce a correct answer in this particular case); we need, instead, to consider if players can gain by deviating unilaterally, *given* beliefs about the others' strategy and the others' beliefs, as demonstrated above.